SOME APPLICATIONS OF
WREATH PRODUCTS OF GROUPS

by

R.G. BURNS

A thesis presented to the
Australian National University
for the degree of
Doctor of Philosophy
in the
Department of Mathematics

Canberra

1966

G51606

✓

STATEMENT

Except where it is indicated otherwise, this thesis is my own work.

R. G. Burns

ACKNOWLEDGEMENTS

I am especially grateful to my supervisor, Professor Hanna Neumann, for her continual help, encouragement, and always constructive and well-deserved criticism.

I thank Professor B.H. Neumann FAA,FRS, for his initial supervision and for suggesting the topic of direct limits of wreath products treated in Part I.

I would also like to thank Dr L.G. Kovács and Dr M.F. Newman for helpful mathematical discussions and my fellow students for stimulating discussions on all subjects.

Finally I thank Mrs Avis Debnam for typing the stencils, and Mr L. Debnam for drawing in the diagrams and checking the stencils.

INDEX

PART II: GROUPS GENERATING CROSS PRODUCT VARIETIES.

LIST OF NOTATION

The following list is of that notation I have used which is not completely standard in modern publications on group theory but which has not been fully introduced in the text.

$\|S\|$	the cardinal of the set S
$S \setminus T$	the set of elements in S but not in the set T
$\emptyset$	the empty set
e	the identity element of the group in question
$gp\{a_1, a_2, \ldots \mid \omega_1 = e, \omega_2 = e, \ldots\}$	the group generated by $a_1, a_2, \ldots,$ with relations $\omega_1 = e, \omega_2 = e, \ldots$.

Let G be a group, let $s, t, \ldots \in G$, and let $S, T, \ldots$ be subsets of G . Let θ be a homomorphism of G .

$sgp\{s, t, \ldots\}$	the subgroup of G generated by $s, t, \ldots$
$sgp\{S, T, \ldots\}$	the subgroup of G generated by $S, T, \ldots$
$sgp\{S, s, T, t, \ldots\}$	$sgp\{S, \{s\}, T, \{t\}, \ldots\}$
E	$sgp\{e\}$
s^t	$t^{-1}st$
$[s, t]$	$s^{-1}t^{-1}st$
$[S, T]$	$sgp\{[s,t] \mid s \in S , t \in T\}$
$[S, t]$	$[S, \{t\}]$
$C_S(T)$	the set of elements in S that centralize T .

S^G	the least normal subgroup of G containing S (i.e. the normal closure of S in G)
$Z(G)$	the centre of G
$\|G\|$	the order of G
$var(G)$	the variety generated by G
$var(Q)$	the variety generated by the set of groups Q
$\ker \theta$	the kernel of θ
$d(G)$	the least number of elements which generate G
$GF(p)$	the field of integers modulo the prime p

CHAPTER 0.

INTRODUCTION.

The thesis is divided into two parts. Although the problems treated in the two parts are in no way connected, the main tool, the wreath product, and to some extent the properties of it that are exploited, are the same. Definitions of the relevant wreath products can be found in the introductions to Chapters 2 and 7 .

The wreath product in its various forms has been used for solving a wide variety of problems (see for instance [1], [6], [21]). Something of this versatility can be seen from the lack of connexion between Parts I and II, or even from differences of application within Part II.

Certain properties of the normal closure of a particular subgroup (the top group) of a wreath product are essential to both parts (Lemma 2.6 and Theorem 7.3). Embedding properties have been used to a lesser degree. The celebrated result of Krasner and Kaloujnine [12] that any extension of a group A by a group B can be embedded in the standard wreath product of A and B is used implicitly in Chapter 4:- a theorem of P. Hall quoted there depends heavily on this fact. In part II, Lemmas 5.7 and 8.2 are statements of embedding properties of different kinds.

In Chapters 4, 6 and 8 the following well-known lemma plays an essential part. We state it here as it is used in both parts of the thesis. The proof is omitted.

The <u>cartesian product</u> of a family $\{G_j\}$ of groups, indexed by a set J say, is the group of all families $\{g_j\}$, $g_j \in G_j$, with multiplication defined by

$$\{g_j\}\{h_j\} = \{g_j h_j\} ; \quad g_j, h_j \in G_j .$$

The terms "embedding" of a group A into a group B, and "monomorphism" of A into B are synonymous.

0.1 <u>Lemma</u>. A group G can be embedded in the cartesian product of the family $\{G_j\}$ if and only if there exists a family $\{N_j\}$, $j \in J$, of normal subgroups of G such that for all $j \in J$, G/N_j can be embedded in G_j, and $\cap_{j\in J} N_j = E$.

In Part I we shall consider ascending sequences of groups. These are a particular case of ascending infinite chains of subgroups of a group, which are important in many parts of the theory of infinite groups.

We are concerned with the following aspects.

A natural question to ask is:- Given two ascending sequences of groups

$$G_1 \leq G_2 \leq \dots ; \quad H_1 \leq H_2 \leq \dots ,$$

such that $G_n \cong H_n$ for $n \geq 1$, when is it true that

$$\cup_{n=1}^{\infty} G_n \cong \cup_{n=1}^{\infty} H_n \ ?$$

A simple sufficient condition is given in Chapter 1: if there exists a sequence of isomorphisms α_n from G_n onto H_n such that, for all n, the restriction of α_{n+1} to G_n is α_n, then the above unions are isomorphic.

Suppose we demand only that for each fixed $n \geq 1$, there exist an isomorphism α_n from G_n onto H_n such that $G_i\alpha_n = H_i$ for $1 \leq i \leq n$. The constructions in Chapter 3 provide examples which satisfy this condition but for which $\cup_{n=1}^{\infty} G_n \not\cong \cup_{n=1}^{\infty} H_n$ (Remark 3.8). However with the added restriction that the automorphism group of G_n be finite for all n, this weakened condition does suffice for isomorphism of the unions (Theorem 3.9).

These sufficient conditions are in general nowhere near being necessary. Examples can easily be concocted with the aid of Theorem 1.11 of Chapter 1, for which, for every n, no isomorphism from G_{n+1} onto H_{n+1} maps G_n onto H_n, yet such that $\cup_{n=1}^{\infty} G_n \cong \cup_{n=1}^{\infty} H_n$. It appears to be difficult to find any comparably general necessary condition.

One might also ask just how many non-isomorphic unions exist for each suitable given sequence of isomorphism classes of groups.

That is, if we are given a sequence $\{G_n \mid n \geq 1\}$ of groups such that G_n can be embedded in G_{n+1} for all n, then how many non-isomorphic groups can be obtained that are unions of ascending sequences of groups isomorphic to the G_n ?

An upper bound for this number can be readily obtained. If for $n \geq 1$, M_n is the set of all monomorphisms of G_n into G_{n+1}, then the cardinal of the cartesian product of the sequence $\{M_n\}$ (that is the set of all sequences $\{\theta_n\}$, $\theta_n \in M_n$) is such an upper bound. For, as will be shown in Chapter 1, any union of a sequence of groups

$$H_1 \leq H_2 \leq \dots ,$$

with $H_n \cong G_n$, $n \geq 1$, is isomorphic to a group called the "direct limit" of $\{G_n\}$ uniquely determined by some sequence $\{\theta_n\}$ of embeddings (Definition 1.2 et seqq.).

It is easy to find examples for which this bound is attained. For instance when G_n is an infinite cycle for every n, then ~~a set of~~ $2^{\aleph_0}$ pairwise non-isomorphic groups that are unions of ascending sequences of infinite cyclic subgroups can be obtained as certain subgroups of the additive group of rational numbers. (See Kurosh [13] Vol. 1, p. 56.) Clearly $2^{\aleph_0}$ is the maximum number possible in this case.

If G_n is finite for all n, then again $2^{\aleph_0}$ is a bound for the number of non-isomorphic direct limits of $\{G_n\}$. In Chapter 3 it is shown that for certain fixed sequences of wreath products, simple variations in the sequence of embeddings give rise to $2^{\aleph_0}$ non-isomorphic direct limits. Since there are only $2^{\aleph_0}$ isomorphism classes of countable groups, this gives us a method of constructing "large" numbers of non-isomorphic locally finite, countable groups.

In Chapter 4 we take G_n to be a certain wreath product of p-power order (p prime) so that the direct limits are locally finite, countably infinite p-groups.

Various infinite p-groups with properties widely different from those of finite p-groups, have been constructed. (See Kurosh [13], Vol. 2, appendix Q.) The ones we obtain have trivial centre and are "verbally complete" according to the following definition due to P. Hall [6].

0.2 <u>Definition</u>. A group G is said to be verbally complete if for every element $g \in G$ and every word $w(x_1,\ldots,x_n)$ (of some auxiliary free group of countably infinite rank) there exist elements $g_1,\ldots,g_n \in G$ such that

$$w(g_1,\ldots,g_n) = g .$$

The fact that we obtain the maximum possible number of $2^{\aleph_0}$ non-isomorphic, locally finite, countably infinite, verbally complete p-groups with trivial centres answers a question asked by P. Hall in the same paper.

Chapter 1 consists of a general introduction to Part I and a theorem (Theorem 1.11) on direct limits of direct powers of a fixed group, which is justified mainly by its applications to Chapter 3. Chapter 2 consists of preliminary definitions and results connected with the wreath product, and Chapters 3 and 4 contain the main results described above.

In Part II entirely different problems are tackled. Before describing them, some of their background should be sketched in. Most of the following definitions and results stated without proof, can be found in some or all of the papers [7], [8], [10], [11], [20], [21].

Let F be a free group on a countably infinite number of free generators. A <u>variety</u> $\underline{\underline{V}}$ corresponding to a set of words $V \subseteq F$, may be defined as the class of all groups G for which the verbal subgroup $V(G)$ is trivial. It can be shown that a variety is equivalently defined as a class of groups closed under the operations of taking homomorphic images, subgroups, and cartesian products. Consequently the variety <u>generated by a set or class of groups</u> is well-defined as the closure of the set or class.

A well-known set of varieties for which the famous "finite basis" problem of B.H. Neumann has been solved (Oates and Powell

[20]) is the set of varieties generated by a single finite group. We call such a variety a <u>Cross variety</u>. That this is equivalent to the usual definition (see Higman [7]) is the result of Oates and Powell [20].

Given a group G, any factor group H/N where $E \leq N \trianglelefteq H \leq G$ is called a <u>factor</u> of G. Provided we do not have both $H = G$ and $N = E$, H/N is said to be a <u>proper factor</u> of G.

The group G is <u>critical</u> if it is not in the variety generated by its proper factors. (Cf. Higman [7].)

By "locally finite" varieties, "nilpotent" varieties etc., we shall mean varieties all of whose groups are locally finite, nilpotent, etc.

One can prove that any critical group is finitely generated, and any locally finite variety is generated by the critical groups contained in it. A Cross variety, which is clearly locally finite, has only finitely many critical groups.

Another important concept (due to Hanna Neumann) is that of a product variety $\underline{\underline{U}}\,\underline{\underline{V}}$ of two varieties $\underline{\underline{U}}$, $\underline{\underline{V}}$. This is defined as the class of all group extensions of groups in $\underline{\underline{U}}$ by groups in $\underline{\underline{V}}$. One has, of course, to prove that this class is a variety.

It is natural to ask for those product varieties which are also Cross varieties. A.L. Šmel'kin [21] has proved that $\underline{\underline{U}}\,\underline{\underline{V}}$

is a Cross variety if and only if $\underline{\underline{U}}$ is nilpotent of exponent m , $\underline{\underline{V}}$ is abelian of exponent n , and $(m,n) = 1$. These particular varieties are the subject of Part II. There they are denoted by $\underline{\underline{N}}\,\underline{\underline{A}}_n$ where $\underline{\underline{N}}$ is nilpotent of class c and exponent m .

The set of varieties generated by a finitely generated group is larger than the set of Cross varieties. If a variety $\underline{\underline{V}}$ is generated by an r-generator group then it is generated by its reduced free group of rank r : that is by the factor group $F_r/V(F_r) = F_r(\underline{\underline{V}})$ where F_r is free of rank r . For such a variety it seems of interest to determine how small r can be made. Denote by $\ell(\underline{\underline{V}})$ the least r such that $F_r(\underline{\underline{V}})$ generates $\underline{\underline{V}}$. In the paper [1] of G. Baumslag, B.H. Neumann, Hanna Neumann and Peter M. Neumann, $\ell(\underline{\underline{V}})$ is found for several kinds of varieties $\underline{\underline{V}}$. Graham Higman [8] has shown that if $\underline{\underline{V}}$ is any nilpotent variety of class c (that is, $\underline{\underline{V}}$ contains nilpotent groups of class precisely c) , then $\ell(\underline{\underline{V}}) \leq c$. It is known (see [1]) that for the variety $\underline{\underline{N}}_c$ of all nilpotent groups of class $\leq c$, $[c/2] \leq \ell(\underline{\underline{N}}_c) \leq c$, where $[c/2]$ denotes the integral part of $c/2$. To date no sharper bound has been obtained.

In contrast with this, the main theorem of Part II (Theorem 5.1) states that if $\underline{\underline{U}}$ is any non-abelian nilpotent variety of class c and exponent m , $\underline{\underline{V}}$ is abelian of exponent n , and

$(m,n) = 1$, then $\ell(\underline{\underline{U}}\,\underline{\underline{V}}) = c$.

In Chapter 5 it is proved that $\ell(\underline{\underline{U}}\,\underline{\underline{V}}) \leq c$. Most of the arguments used there are due to L.G. Kovács.

In Chapter 6 we determine the subvariety lattices of varieties $\underline{\underline{U}}\,\underline{\underline{V}}$ of the above form for $\underline{\underline{U}}$ nilpotent of class 2 and certain pairs of exponents m,n . It turns out that these particular varieties, for which $\ell(\underline{\underline{U}}\,\underline{\underline{V}}) = 2$, each contain a critical group which is a strictly 3-generator group. This provides a limitation (in a certain weak sense - see the introduction to Chapter 6) to a remark of Hanna Neumann (Theorem 6.2) that a variety generated by a single k-generator group is not generated by its (k - 1) - generator groups. I do not touch the more general question of whether for each positive integer k there exists a variety generated by its k-generator groups, and also by a set of critical groups some or all of which require strictly more than k generators.

In Chapter 7, a characterisation of the normal closures of subgroups of the "top group" of a verbal wreath product is obtained, and this is applied in Chapter 8 to obtain $\ell(\underline{\underline{U}}\,\underline{\underline{V}}) \geq c$ for product varieties $\underline{\underline{U}}\,\underline{\underline{V}}$ of the above form.

We conclude the introduction with a description of one of the principal tools used in Part II.

Write $\Pi^*_{j\in J}\, G_j$ for the free product of the family of groups $\{G_j\}$ indexed as before by J. (For a definition, see for example Kurosh [13] , Vol. 2.). The cartesian subgroup $C = C(\Pi^*_{j\in J})G_j$, of the free product, is the ^normal^ subgroup generated by all commutators $[g_{j_1}, g_{j_2}]$ where $g_{j_1} \in G_{j_1}$, $g_{j_2} \in G_{j_2}$, and $j_1 \neq j_2$. Loosely speaking, the cartesian subgroup is the normal subgroup that has to be factored out of the free product to give the direct product of the family $\{G_j\}$.

Suppose V is a set of words in the auxiliary free group F . The following definition is due to S. Moran [14] .

0.3 Definition. The verbal $\underline{\underline{V}}$-product of the family $\{G_j\}$ is the factor group

$$\Pi^{\underline{\underline{V}}}_{j\in J} G_j = (\Pi^*_{j\in J} \; G_j) \Big/ (V(\Pi^*_{j\in J} G_j) \cap C) .$$

S. Moran [14] has shown that this product is associative in the obvious sense and that it shares other properties with the free and direct products. In fact the direct product of $\{G_j\}$ may be looked upon as the verbal $\underline{\underline{A}}$-product of $\{G_j\}$ where A is the derived group of F , and $\underline{\underline{A}}$ is the variety of all abelian groups. We still denote it by $\Pi^{\times}_{j\in J} G_j$. (This notation will also be used for the "internal" direct product of a set of subgroups of a group.)

We shall in Part II sometimes regard the G_j as subgroups of the verbal product under consideration. That is, we shall not distinguish between G_j and $(G_j.(V(\Pi^*_{j\in J}G_j) \cap C))/(V(\Pi^*_{j\in J}G_j) \cap C)$.

The distinction will always be clear from the context.

PART I

ASCENDING SEQUENCES OF GROUPS

CHAPTER 1.

DIRECT LIMITS AND DIRECT POWERS

Introduction: direct limits.

It is often convenient to express a group as a union of an ascending sequence of subgroups

1.1 $$H_1 \leq H_2 \leq \dots ,$$

or a group may occur as such a union. For example, a countable locally finite group is the union of an ascending sequence of finite groups; a ZA-group of class ω , the first infinite ordinal, is the union of its upper central series; a group of type p^∞ , p , p prime , is the union of p-power cycles. (See Kurosh [13].)

Suppose more generally that a sequence $\{G_n \mid n = 1,2,\dots\}$ of groups and a sequence $\{\theta_n \mid G_n\theta_n \leq G_{n+1}\}$ of monomorphisms are given. Then the following construction (Kurosh [13], vol. 1) yields in the natural way a union of an ascending sequence of groups $\{\overline{G}_n\}$ for which $\overline{G}_n \cong G_n$.

1.2 <u>Definition</u>. The direct limit L of the sequence $\{G_n\}$ with embeddings $\{\theta_n\}$ is the group consisting of all threads; that is sequences

$$g_k, g_{k+1}, \dots ,$$

subject to the following conditions:

(1) $k \geq 1$;

(2) $g_{k+i} \in G_{k+i}$, $i = 0,1,\ldots$;

(3) $g_{k+i}\theta_{k+i} = g_{k+i+1}$, $i = 0,1,\ldots$;

(4) if $k > 1$ then g_k is the image under θ_{k-1} of no element of G_{k-1} .

The multiplication of threads

$$\gamma = g_k, g_{k+1}, \ldots ,$$

and

$$\gamma' = g'_\ell, g'_{\ell+1}, \ldots ,$$

is given by

$$\gamma\gamma' = g_m g'_m, \; g_{m+1}g'_{m+1}, \ldots ,$$

where $m = \max(\ell, k)$ if $k \neq \ell$. If $k = \ell$, the sequence $g_\ell g'_\ell, g_{\ell+1}g'_{\ell+1}, \ldots$, is completed to a thread by adding, if necessary, suitable elements at the beginning to ensure that (4) is satisfied.

Then it is easy to check that this product is indeed a thread and that this multiplication defines a group. (See Kurosh [13], Vol 1.)

If H is a subgroup of G_n for some n , denote by $\overline{H}$

that group of threads which contain an element of H as a term. Similarly, if $g \in G_n$ write $\overline{g}$ for that thread in L containing g as a term. Then $\overline{G}_n \cong G_n$ and L is the union of the ascending sequence

$$\overline{G}_1 \leq \overline{G}_2 \leq \ldots$$

Now if this construction is carried out with the groups H_n of the sequence 1.1 in place of the G_n, taking each θ_n as the identity mapping, then the mapping

$$h \longrightarrow \overline{h}\ , \quad h \in \bigcup_{n=1}^{\infty} H_n\ ,$$

is an isomorphism from $\bigcup_{n=1}^{\infty} H_n$ onto $\bigcup_{n=1}^{\infty} \overline{H}_n$, the group obtained by forming threads.

More generally, suppose we are given two finite or infinite sequences $\{G_n\}$ and $\{K_n\}$ of groups. The sequences are assumed to be both infinite or both finite of the same length. Let $\{\theta_n \mid G_n\theta_n \leq G_{n+1}\}$ and $\{\varphi_n \mid K_n\varphi_n \leq K_{n+1}\}$ be sequences of monomorphisms. The following definition is a generalization of the one given by Philip Hall [6] for the case when $\{G_n\}$ and $\{K_n\}$ have length 2 .

1.3 Definition. The sequences $\{\theta_n\}$ and $\{\varphi_n\}$ are said to be of the same type if there exists a sequence $\{\alpha_n\}$ of isomorphisms α_n from G_n onto K_n, such that

1.4 $$\theta_n \alpha_{n+1} = \alpha_n \varphi_n \quad .$$

That is to say $\{\theta_n\}$ and $\{\varphi_n\}$ are of the same type if there exists a sequence $\{\alpha_n\}$ of isomorphisms such that for each n the diagram

$$\begin{tikzcd}
G_n \arrow[r, "\theta_n"] \arrow[d, "\alpha_n"'] & G_{n+1} \arrow[d, "\alpha_{n+1}"] \\
K_n \arrow[r, "\varphi_n"'] & K_{n+1}
\end{tikzcd}$$

is commutative.

Then the following lemma is simple to prove.

1.5 Lemma. If two sequences of embeddings are of the same type then the direct limits constructed from them are isomorphic.

Proof. Let $\{G_n\}$, $\{\theta_n\}$ (giving rise to direct limit L_1 say) and $\{K_n\}$, $\{\varphi_n\}$ (with direct limit L_2) be as in Definition 1.3 and suppose a sequence $\{\alpha_n\}$ of isomorphisms from G_n onto K_n exists satisfying 1.4. We shall now construct an isomorphism α between L_1 and L_2.

Let $\overline{g}_n \in L_1$ where $g_n \in G_n$. Since L_1 is the union of the $\overline{G}_i$, every element of L_1 may be written in the form $\overline{g}_n$ for some positive integer n. Set

1.6 $$\overline{g}_n\alpha = \overline{g_n\alpha_n} \in L_2 .$$

To verify that α is a mapping we prove that $\overline{g'_m}\alpha = \overline{g}_n\alpha$ implies that $\overline{g'_m} = \overline{g}_n$ for $g'_m \in G_m$, $m \geq n$. Suppose

$$\overline{g'_m}\alpha = \overline{g}_n\alpha$$

Then by 1.6,

$$\overline{g'_m\alpha_m} = \overline{g_n\alpha_n} ,$$

which implies by Definition 1.3 that

$$g'_m\alpha_m = g_n\alpha_n\varphi_n \ldots \varphi_{m-1} .$$

Applying 1.4 $m - n$ times, we obtain

$$g'_m\alpha_m = g_n\theta_n \ldots \theta_{m-1}\alpha_m ,$$

and since α_m is an isomorphism,

$$g'_m = g_n\theta_n \ldots \theta_{m-1} ;$$

that is

$$\overline{g'_m} = \overline{g}_n .$$

The mapping α is clearly onto L_2. It is (1,1) since the relation $\alpha' : L_2 \longrightarrow L_1$ defined by

$$\bar{k}_n\alpha' = \overline{k_n\alpha_n^{-1}}, \quad k_n \in K_n ,$$

is a mapping by the same argument as above, and is thus the inverse mapping of α.

To prove that α is homomorphic it suffices to show that

$$(\bar{g}_m'.\bar{g}_n)\alpha = (\bar{g}_m'\alpha).(\bar{g}_n\alpha)$$

where $g_m' \in G_m$ and $m \geq n$. Now by 1.3,

$$\begin{aligned}(\bar{g}_m'.\bar{g}_n)\alpha &= \overline{g_m'(g_n\theta_n\ldots\theta_{m-1})}\alpha \\ &= \overline{(g_m'(g_n\theta_n\ldots\theta_{m-1}))\alpha_m} \quad \text{(by 1.6)} \\ &= \overline{(g_m'\alpha_m).(g_n\alpha_n\varphi_n\ldots\varphi_{m-1})} \quad \text{(by 1.4)} \\ &= \overline{g_m'\alpha_m}.\overline{g_n\alpha_n} \quad \text{(by 1.3)} \\ &= \bar{g}_m'\alpha.\bar{g}_n\alpha \quad \text{(by 1.6)} .\end{aligned}$$

This completes the proof of the lemma.

The following corollary is almost self-evident.

1.7 Corollary. Suppose $\{G_n\}$ is an infinite sequence of groups and $\{\theta_n \mid G_n\theta_n \leq G_{n+1}\}$ a sequence of monomorphisms giving rise to direct limit L. If $\{G_{i_n}\}$ is an infinite subsequence of $\{G_n\}$ and for $n = 1,2,\ldots$, $\Psi_n = \theta_{i_n}\theta_{i_n+1}\cdots\theta_{i_{n+1}}$, then the direct limit L^* formed from $\{G_{i_n}\}$ and $\{\Psi_n\}$ is isomorphic to L.

Proof. The proof is briefly as follows. We have $L = \cup_{n=1}^{\infty} \overline{G}_{i_n}$. Let φ_n be the identity map on $\overline{G}_{i_n}$, $n = 1,2,\ldots$. If, for $n = 1,2,\ldots$, α_n is the isomorphism from $\overline{G}_{i_n}$ onto G_{i_n} defined by

$$\overline{g}_{i_n}\alpha_n = g_{i_n}$$

where $g_{i_n} \in G_{i_n}$, then the condition 1.4 is readily seen to be satisfied and hence $L \cong L^*$ by Lemma 1.5.

Theorem 1.11 below and its proof yield examples of a sequence of groups $\{G_n\}$ with two sequences $\{\theta_n \mid G_n\theta_n \leq G_{n+1}\}$ and $\{\varphi_n \mid G_n\varphi_n \leq G_{n+1}\}$ with no non-empty subsequences of the same type, yet giving rise to isomorphic direct limits. Thus the condition that two sequences of embeddings be of the same type, though sufficient for isomorphism of the direct limits by Lemma 1.5, is by no means necessary in general. On the other hand it will appear in Chapter 3 (Remark 3.8) that no obvious weakening of this condition is possible in general.

Since the problem of finding some reasonably general necessary condition for isomorphism of direct limits seems unattackable even when the sequence of groups is fixed, we consider, as outlined in Chapter 0, some special kinds of sequences of groups and embeddings. By fixing the sequence of groups but varying the embeddings, we construct large numbers of non-isomorphic countably infinite groups with unusual properties (Chapters 2, 3, 4).

Direct limits of direct powers of a fixed group.

The direct limits of wreath products which we shall construct later, contain as subgroups certain direct limits of finite direct powers of a fixed finite group. Since these subgroups will be of interest to us and since a sequence of finite increasing direct powers of a fixed group G is an obvious simple example of a sequence of groups possessing natural embeddings at each step, we start by investigating the direct limits obtainable from sequences such as these (in the case of finite G) with suitable restrictions on the embeddings.

Before we can give the main theorem a few definitions are required. Let S be any non-empty set. Then in standard notation, G^S denotes the group of all functions from S into G with the product fh of functions f, h defined at each argument $s \in S$ by

$$(fh)(s) = f(s)h(s) .$$

If $S = \{1,\dots,n\}$ then it is usual to write G^n for G^S. The group G^S is called a <u>cartesian power</u> of G: it is isomorphic to the cartesian product of $|S|$ isomorphic copies of G.

The <u>support</u> $\mathrm{supp}(F)$ of a subset F of G^S is the subset of all those elements $s \in S$ such that for some $f \in F$, $f(s)$ is not the unit element of G. Similarly we speak of the support of a function $f \in F$ and write $\mathrm{supp}(f)$. The subgroup of G^S consisting of all functions with finite support is called a <u>direct power</u> of G, denoted by $G^{(S)}$. Clearly if S is finite, $G^{(S)} = G^S$.

If $\emptyset \neq S_1 \subseteq S$ then G^{S_1} will denote the subgroup of all functions $f \in G^S$ with $\mathrm{supp}(f) \subseteq S_1$, and $G[S_1]$ will mean the subgroup of all functions constant on S_1 and taking the value e outside S_1. If $g \in G$ then $g[S_1]$ will denote that function in $G[S_1]$ taking the value g on S_1. In particular, $G[S]$ is called the <u>diagonal</u> of G^S and $G[\{s\}] = G[s]$, $s \in S$, the sth <u>co-ordinate subgroup</u>. Similarly we adopt the notation $g[s]$ for that function f in $G[s]$ for which $f(s) = g$: the square brackets are used to avoid the misinterpretation of $g[s]$ as the value of some function at s. Obviously G^S is the cartesian product of its co-ordinate subgroups.

For the rest of this chapter G will denote a finite group.

We are concerned with direct limits of sequences of finite direct powers of G, with fairly natural restrictions on the embeddings allowed. More explicitly, we take a sequence of non-empty finite sets

$$S_1, S_2, \ldots ,$$

with $S_i = \{s_i^1, \ldots, s_i^{n_i}\}$ (where the superscripts $1, \ldots, n_i$ are used to distinguish the elements of S_i) so that $|S_i| = n_i$, $i = 1, 2, \ldots$, where $n_i \longrightarrow \infty$ monotonically with i. Then the sequence of groups we consider, is

$$G^{S_1}, G^{S_2}, \ldots$$

with monomorphisms $\theta_i : G^{S_i} \longrightarrow G^{S_{i+1}}$, $i = 1, 2, \ldots$, restricted by the following two requirements.

$$\text{1.8} \quad \begin{cases} \text{For each } s_i^j \in S_i, \ G[s_i^j]\theta_i \text{ is to be a subdirect product} \\ \text{of a part product of } G^{S_{i+1}}, \text{ for all } i . \end{cases}$$

(That is to say that the projection of $G[s_i^j]\theta_i$ on each co-ordinate subgroup of $G^{S_{i+1}}$ is to be the whole of that co-ordinate subgroup or else trivial.)

The second, less natural, requirement needs a preliminary definition. Given a set S, the set of all subsets (the power set) of S is denoted by $\mathcal{P}(S)$. A mapping τ_i ~~form~~ from $\mathcal{P}(S_i)$ into $\mathcal{P}(S_{i+1})$ is defined for each i as follows. Let $S_i' \subseteq S_i$; then set

$$S_i'\tau_i = \mathrm{supp}(G^{S_i'}\theta_i) \ .$$

Then the second requirement is that

$$1.9 \quad \begin{cases} \text{for all } i \geq 1 \text{ and all } s_i^j \in S_i \text{ there exist a positive} \\ \text{integer } k(i,j) \text{ such that} \\ \qquad |\{s_i^j\}\tau_i \ldots \tau_{i+k(i,j)-1}| > 1 \ . \end{cases}$$

Let L denote the direct limit of $\{G^{S_i}\}$ with a fixed sequence $\{\theta_i\}$ of monomorphisms satisfying 1.8 and 1.9. Theorem 1.11 will show that, up to isomorphism, there are at most two possibilities for L , and these can be simply defined as subgroups of G^I , where I is the set of positive integers.

This fact that, assuming 1.9, at most two non-isomorphic direct limits can occur is the one that is relevant to the later chapters. Without the condition 1.9, the resulting direct limits can still be described as subgroups of G^I in a way similar to that emerging from the proof of Lemma 1.13 below: this indicates that the number of non-isomorphic direct limits then arising may not even be countable, but I have not been able to prove anything definite.

In order to state the theorem (with the conditions 1.8

and 1.9), we define one particular subgroup of G^I. This requires again some preliminary definitions.

A partition P of a set S is a set P of subsets of S whose union is S and such that any two distinct subsets intersect in the empty set. Given two partitions P_1 and P_2 of S, we say that P_2 is a sub-partition of P_1 if P_2 is obtained from P_1 by partitioning the subsets in P_1.

We now define, by induction on i, a sequence of partitions of I into infinite subsets :

1.10 $$P(1), P(2), \ldots, P(i), \ldots ,$$

such that $P(i+1)$ is a sub-partition of $P(i)$.

Set $P(1) = \{I\} = \{P_{1,1}\}$ say, and assume inductively that $P(i)$ has been defined:

$$P(i) = \{P_{i,1}, \ldots, P_{i,2^{i-1}}\} .$$

Then for each $j \in \{1, \ldots, 2^{i-1}\}$ let

$$\{P_{i+1,j}, P_{i+1,j+2^{i-1}}\}$$

be any partition of $P_{i,j}$ into two infinite subsets.
Then $P(i+1)$ is the partition

$$\{P_{i+1,1}, \ldots, P_{i+1,2^i}\}$$

of I into 2^i infinite subsets.

Evidently if S' and S'' are non-empty subsets of a set

S such that $S' \cap S'' = \phi$, then as subgroups of G^S, $G^{S'}$ and $G^{S''}$ generate their direct product.

With this in mind we can write down the subgroup L_0 of G^I that we are concerned with, as follows:

$$L_0 = \operatorname{sgp}\{\Pi^{2^{i-1}\times}_{j=1} G[P_{i,j}] \mid i \in I\} .$$

Note that, up to isomorphism, L_0 is independent of the particular partitions taken, provided they are formed according to the definition: that is, I is partitioned into two infinite subsets, these are treated likewise and so on. For if $P'(i) = \{P'_{i,j} \mid 1 \leq j \leq 2^{i-1}\}$ is similarly defined, then there is a $(1,1)$ mapping of I onto itself which maps $P_{i,j}$ onto $P'_{i,j}$. Then clearly the mapping

$$g[P_{i,j}] \longrightarrow g[P'_{i,j}], \quad g \in G ,$$

defines an isomorphism between L_0 and the group obtained in the same way from $\{P'(i)\}$. This fact will be used subsequently.

We deduce briefly a few properties of L_0. Since $P(i+1)$ is a sub-partition of $P(i)$ for all $i \in I$, it follows that the sequence $\{\Pi^{2^{i-1}\times}_{j=1} G[P_{i,j}]\}$ is ascending and hence that

$$L_0 = \cup^{\infty}_{i=1}(\Pi^{2^{i-1}\times}_{j=1} G[P_{i,j}]) .$$

For each $i \geq 2$ consider the set $\{P_{i,j} \mid j \text{ odd}, 1 \leq j \leq 2^{i-1}\}$. The notation is such that this is, for each i, a partition of $P_{2,1}$; call it $P_1(i)$. Then we also have that $P_1(i+1)$ is a sub-partition of $P_1(i)$. Similarly, for each $i \geq 2$, $\{P_{i,j} \mid j \text{ even}, 1 \leq j \leq 2^{i-1}\}$ is a partition, $P_2(i)$ say, of $P_{2,2}$, and $P_2(i+1)$ is a sub-partition of $P_2(i)$. Write

$$L_1 = \cup_{i=2}^{\infty}(\Pi_{j=1}^{2^{i-2}\times} G[P_{i,2j-1}]) \; ; \; L_2 = \cup_{i=2}^{\infty}(\Pi_{j=1}^{2^{i-2}\times} G[P_{i,2j}]) \; .$$

From the remark above, one has in particular that $L_1 \cong L_2 \cong L_0$. But also, L_1 and L_2 have supports intersecting trivially. Hence $\operatorname{sgp}\{L_1, L_2\} = L_1 \times L_2$; moreover $L_0 = \operatorname{sgp}\{L_1, L_2\}$. Thus $L_0 \cong L_0 \times L_0$ whence it follows that $L_0 \cong L_0^i$ for all $i \in I$.

However it is not true in general that $L_0 \cong L_0^{(I)}$. For suppose G has trivial centre. Then the centralizer of any finite set of elements of $L_0^{(I)}$ is infinite (since any finite set of elements must have finite support) whereas the finite subgroup $G[P_{1,1}] = G[I]$ of L_0 has trivial centralizer. Thus $L_0 \ncong L_0^{(I)}$ in this case.

We can now formulate the theorem.

1.11 <u>Theorem</u>. With restrictions 1.8 and 1.9 and G finite, the direct limit L of $\{G^{S_i}\}$ is isomorphic either to L_o or to $L_o^{(I)}$.

<u>Remark</u>. This contrasts with the main theorem of Chapter 3 (Theorem 3.1) where continuously many non-isomorphic direct limits are obtained from a fixed sequence of finite groups by varying the sequence of embeddings in a correspondingly simple way.

The theorem and its proof find applications in Chapter 3 (Remarks 3.2 and 3.8).

<u>Proof of the theorem</u>. If G is abelian it is not difficult to show that any direct limit of the sequence $\{G^{S_i}\}$ with embeddings restricted by 1.8 and 1.9, is isomorphic to $G^{(I)}$ (as is L_o) . It is therefore assumed that G is non-abelian, in which case, by 1.8, we have for each $i \in I$,

1.12 $$\mathrm{supp}(G[s_i^j]\theta_i) \cap \mathrm{supp}(G[s_i^k]\theta_i) = \emptyset \ ;$$
$$j \neq k \ ; \ \ j,k \in \{1,\ldots,n_i\} \ .$$

The proof of the theorem splits into two parts. We frame the first part as a lemma.

1.13 <u>Lemma</u>. If in addition to the hypotheses of Theorem 1.11 it is assumed that

1.13.i $$S_i\tau_i \ (= \operatorname{supp}(G^{S_i}\theta_i)) = S_{i+1} \ ,$$

then L is isomorphic to L_o .

Proof. The proof consists first of constructing an ascending sequence of subgroups of G^I with its sequence of embeddings of the same type as $\{\theta_i\}$, and second, of showing that the union of this ascending sequence of subgroups is isomorphic to L_o .

We again form a sequence of partitions of I :

$$Q(1), Q(2), \ldots, Q(i), \ldots,$$

where $Q(i+1)$ is a sub-partition of $Q(i)$. We shall define $Q(i)$ to be a partition $\{Q_{i,1}, \ldots, Q_{i,n_i}\}$ of I into n_i infinite subsets, by induction on i .

Define $Q(1) = \{Q_{1,1}, \ldots, Q_{1,n_1}\}$ to be any partition of I into n_1 infinite subsets, and assume inductively that $Q(i) = \{Q_{i,1}, \ldots, Q_{i,n_i}\}$ has been defined.

Suppose that for each j in $1 \leq j \leq n_i$,

$$\{s_i^j\}\tau_i = \{s_{i+1}^{k_1}, \ldots, s_{i+1}^{k_{m(i,j)}}\} \ .$$

By 1.12, if $j \neq k$ then $\{s_i^j\}\tau_i \cap \{s_i^k\}\tau_k = \phi$. Let

$\{Q_{i+1,k_1},\ldots,Q_{i+1,k_{m(i,j)}}\}$ be any partition of $Q_{i,j}$ into $m(i,j)$ infinite subsets. By the assumption 1.13.1 every $s^k_{i+1} \in S_{i+1}$ belongs to some $\{s^j_i\}\tau_i$ (and by 1.12 to only one such subset of S_{i+1}) and so $Q_{i+1,k}$ is defined for all k in $1 \leq k \leq n_{i+1}$, and $Q_{i+1,k} \cap Q_{i+1,j} = \emptyset$ for $j \neq k$. Hence $Q(i+1)$ is defined.

Since $Q(i+1)$ is a sub-partition of $Q(i)$, the following sequence of subgroups of G^I is ascending :

$$\{\Pi^{n_i}_{j=1}{}^{\times}\, G[Q_{i,j}]\} = \{K_i\} \text{ say.}$$

If S' is any non-empty subset of a set S, then write $\mu(S')$ for the projection of G^S onto its direct factor $G^{S'}$.

Using this, we define for each $i \in I$, an isomorphism α_i from G^{S_i} onto K_i. Set

$$g[s^j_1]\alpha_1 = g[Q_{1,j}] \ , \quad 1 \leq j \leq n_1 \ , \quad g \in G \ .$$

For $i > 1$, write for each j, $1 \leq j \leq n_1$, and each $s^k_i \in \{s^j_1\}\tau_1\ldots\tau_{i-1}$,

$$g[s^j_1]\theta_1\ldots\theta_{i-1}\,\mu(\{s^k_i\})\alpha_i = g[Q_{i,k}] \ , \quad g \in G \ .$$

The assumption 1.13.1 implies that every s_i^k , $1 \le k \le n_i$ belongs to $\{s_1^j\}\ \tau_1 \ldots \tau_{i-1}$ for some j . Hence elements of the form $g[s_1^j]\theta_1 \ldots \theta_{i-1}\ \mu(\{s_i^k\})$ generate G^{S_i} . Clearly α_i is an isomorphism from G^{Si} onto K_i .

Let φ_i be the identity mapping of K_i onto itself. Then we prove that

1.13.2 $$\theta_i\alpha_{i+1} = \alpha_i\varphi_i \ , \quad i \in I \ ,$$

whence it follows by Definition 1.3 and Lemma 1.5 that $L \cong \cup_{i=1}^{\infty} K_i$.

We first prove 1.13.2 for $i = 1$. It suffices to show that for each $s_1^j \in S_1$,

1.13.3 $$g[s_1^j]\theta_1\alpha_2 = g[s_1^j]\alpha_1\varphi_1 \ .$$

Now

$$g[s_1^j]\theta_1 = g[s_1^j]\theta_1\ \mu(\{s_1^j\}\tau_1) = \Pi_{t=1}^{m}(g[s_1^j]\theta_1\ \mu(\{s_2^{k_t}\})) \ ,$$

where $\{s_1^j\}\tau_1 = \{s_2^{k_1},\ldots,s_2^{k_m}\}$. Thus by the definition of α_2 ,

$$g[s_1^j]\theta_1\alpha_2 = \Pi_{t=1}^{m}(g[s_1^j]\theta_1\ \mu(\{s_2^{k_t}\})\alpha_2)$$

$$= \Pi_{t=1}^{m}\ g[Q_{2,k_t}] = g[\cup_{t=1}^{m}\ Q_{2,k_t}] \ .$$

But, $\cup_{t=i}^{m} Q_{2,k_t} = Q_{1,j}$. Thus the left hand side of 1.13.3 is $g[Q_{1,j}]$. The right hand side of 1.13.3 is the same as $g[s_1^j]\alpha_1$ since φ_1 is an identity mapping ; and by the definition of α_1 ,

$$g[s_1^j]\alpha_1 = g[Q_{1,j}] .$$

We now prove 1.13.2 for $i > 1$. Since, as remarked above, G^{S_i} is generated by elements of the form $g[s_1^j]\theta_1 \dots \theta_{i-1}\mu(\{s_i^k\})$ for $i > 1$, it is enough to prove that

1.13.4 $$g[s_1^j]\theta_1 \dots \theta_{i-1}\mu(\{s_i^k\})\theta_i\alpha_{i+1}$$

$$= g[s_1^j]\theta_1 \dots \theta_{i-1}\mu(\{s_i^k\})\alpha_i$$

By the definition of τ_i , the left hand side of 1.13.4 is

$$g[s_1^j]\theta_1 \dots \theta_i\mu(\{s_i^k\}\tau_i)\alpha_{i+1} ,$$

and if $\{s_i^k\}\tau_i = \{s_{i+1}^{k_1}, \dots, s_{i+1}^{k_m}\}$, this becomes

$$\Pi_{t=1}^{m} (g[s_1^j]\theta_1 \dots \theta_i\mu(\{s_{i+1}^{k_t}\})\alpha_{i+1}) ,$$

$$= \Pi_{t=1}^{m} g[Q_{i+1,k_t}] \text{ by the definition of } \alpha_{i+1} .$$

But as for the case $i = 1$, this is the same as $g[Q_{i,k}]$, which is equal to the right hand side of 1.13.4 by the definition of α_i!

To complete the proof of the lemma it remains to show that $\cup_{i=1}^{\infty} K_i$ is isomorphic to L_0.

We define a third and final partition of I:

$$R(1), R(2), \dots, R(i), \dots ,$$

where $R(i) = \{R_{i,1}, \dots, R_{i,2^{i-1}}\}$. We use induction on i.

Set $R(1) = \{I\} = \{R_{1,1}\}$. Let r be the first integer such that $n_r > 1$, where n_r is as before the number of elements in S_r. Then write

$$R_{2,1} = Q_{r,1} \; ; \quad R_{2,2} = \cup_{t=2}^{n_r} Q_{r,t} .$$

Assume inductively that $R(i)$ is defined, and that $r(i,j)$ is the first integer such that $R_{i,j}$, $1 \leq j \leq 2^{i-1}$, is expressible as a union

$$\cup_{t=1}^{n(i,j)} Q_{r(i,j),k_t} ,$$

where the k_t also depend on i, j.

Then if $n(i,j) > 1$, set

$$R_{i+1,j} = Q_{r(i,j),k_1} \; ; \quad R_{i+1,j+2^{i-1}} = \cup_{t=2}^{n(i,j)} Q_{r(i,j),k_t}$$

If $n(i,j) = 1$, let $\ell(i,j)$ be the first integer such that $Q_{r(i,j),k_1}$ is expressible as a union

$$\cup_{t=1}^{n'(i,j)} Q_{r(i,j)+\ell(i,j),k'_t} \, , \quad \text{with } n'(i,j) > 1 \; .$$

That is, $\ell(i,j)$ is the first integer such that

$$|\{s^{k_1}_{r(i,j)}\}\tau_{r(i,j)}\cdots\tau_{r(i,j)+\ell(i,j)-1}| > 1 \; .$$

Condition 1.9 ensures that $\ell(i,j)$ exists.

Then set

$$R_{i+1,j} = Q_{r(i,j)+\ell(i,j),k'_1} \; ,$$

and

$$R_{i+1,j+2^{i-1}} = \cup_{t=2}^{n'(i,j)} Q_{r(i,j)+\ell(i,j),k'_t} \; .$$

Thus $\{R_{i+1,j}, R_{i+1,j+2^{i-1}}\}$ is a partition of $\{R_{i,j}\}$ into two infinite subsets. Since the sequence $\{R(i)\}$ was constructed in the same manner as $\{P(i)\}$ (1.10), we have, by a previous remark (p. 25)

$$L_0 \cong \cup_{i=1}^{\infty} \Pi_{j=1}^{2^{i-1}} G[R_{i,j}] \; .$$

It follows (by a straightforward induction on i) from the way $\{R(i)\}$ has been defined that for each i,j , $Q_{i,j} = R_{k,\ell}$ for some k,ℓ . Hence

$$\cup_{i=1}^{\infty}\left(\Pi_{j=1}^{2^{i-1}} G[R_{i,j}]\right) = \cup_{i=1}^{\infty}\left(\Pi_{j=1}^{2^{i-1}} G[Q_{i,j}]\right) = \cup_{i=1}^{\infty} K_i$$

But $\cup_{i=1}^{\infty} K_i \cong L$ and the proof of the lemma is complete.

Now relax the condition 1.13.1 of Lemma 1.13. Write $T_1 = S_1$ and for $i > 1$ let T_i be the (possibly empty) subset $S_i \setminus S_{i-1}\tau_{i-1}$. For each $i \in I$ consider the sequence

$$\ddot{T}_i, \ddot{T_i\tau_i}, \ldots, \dddot{T_i\tau_i\cdots\tau_{i+k-1}}, \ldots,$$

where $\dddot{T_i\tau_i\ldots\tau_{i+k-1}} \subseteq S_k$.

If $i > j$ we clearly have by 1.12,

1.14 $$\dot{T}_i\tau_i\cdots\tau_{i+k-1} \cap \dot{T}_j\tau_j\cdots\tau_i\cdots\tau_{i+k-1} = \emptyset ,$$

for all $k \in I$. Write, for all $i \in I$,

$$S_{i,0} = T_i$$

and

$$S_{i,k} = T_i\tau_i\cdots\tau_{i+k-1} , \quad k \in I .$$

Then 1.14 can be rewritten as

$$1.15 \qquad \operatorname{supp}(G^{S_{i,k}}) \cap \operatorname{supp}(G^{S_{j,i-j+k}}) = \emptyset ,$$

where $G^{S_{i,k}}$ and $G^{S_{j,i-j+k}}$ are regarded as subgroups of G^{S_k}.

Let $\theta_{i,o}$ be the restriction of θ_i to $G^{T_i} = G^{S_{i,o}}$, and $\theta_{i,k}$ be the restriction of θ_k to $G^{S_{i,k}}$. Then for $i \in I$ denote by L_i the direct limit of the sequence $\{G^{S_{i,k}} \mid k = 0,1,\ldots\}$ with monomorphisms $\theta_{i,k}$, $k = 0,1,\ldots$. Obviously $L_i \leq L$, and all the L_i generate L.

It follows from 1.15 and Definition 1.2 that when $i \neq j$,

$$[L_i,L_j] = E ,$$

and

$$L_i \cap \operatorname{sgp}\{L_j \mid j \neq i\} = E .$$

Thus L is the direct product of its subgroups L_i; $L = \Pi_{i=1}^{\infty} L_i$.

Now since each $\theta_{i,k}$ is a restriction of θ_k, the sequence $\{\theta_{i,k} \mid k = 1,2,\ldots\}$ satisfies for each fixed i the conditions 1.8 and 1.9. Clearly $\{\theta_{i,k}\}$ also satisfies the condition 1.13.1 of Lemma 1.13. Hence if $L_i \neq E$ then $L_i \cong L_0$ and the proof of Theorem 1.11 is complete.

Finally we remark that little group structure has been used to prove Theorem 1.11 and hence that the theorem holds for more general algebraic systems (e.g. finite semigroups).

CHAPTER 2.

WREATH PRODUCTS AND DIRECT LIMITS.

Introduction.

We turn to direct limits of sequences of wreath products. One reason for considering such sequences is their high yield of non-isomorphic direct limits per fixed sequence of groups (see Theorem 3.1 of Chapter 3). In contrast with Theorem 1.11 of Chapter 1 we obtain in Chapter 3 continuously many non-isomorphic direct limits for each of certain fixed sequences of wreath products, although the restrictions on the embeddings correspond closely to the restrictions imposed in the case of sequences of direct powers.

Another reason is that many of the direct limits obtained possess a surprising similarity without being isomorphic (see Remark 3.8). They provide counterexamples to a weakening of the hypothesis of Lemma 1.13.

Thirdly, P. Hall [6] has shown that, in certain circumstances, direct limits of sequences of wreath products are verbally complete. (See Definition 0.2 and Theorem 4.2.) The result obtained in Chapter 4, which uses this fact, answers a question posed by P. Hall in the paper [6] .

This chapter is devoted to preliminaries; we prove several more-or-less well-known lemmas on the permutational wreath product of two permutation groups, and then go on to make precise which direct limits of wreath products are to be considered. A final lemma on these direct limits then prepares us for the main theorem and its proof which are then the subject of Chapter 3.

<u>The wreath product</u>.

Some of our definitions and notations are similar to those of P. Hall [6].

Let X and Y be arbitrary sets and let G and H be groups of permutations of X and Y respectively. Let $P = X \times Y$ denote the set of all ordered pairs (x,y) with $x \in X$, $y \in Y$. For $\alpha \in G$ and $y \in Y$ define the permutation $\alpha(y)$ of P by the rules:

$$(x,y)\alpha(y) = (x\alpha,y) \ ; \quad (x,y_1)\alpha(y) = (x,y_1) \quad \text{if} \quad y_1 \neq y \ .$$

Then the mapping $\alpha \longrightarrow \alpha(y)$ for all $\alpha \in G$, is an isomorphism from G onto a group $G(y)$, and the product

$$\Pi G(y) = \Pi_{y \in Y} G(y)$$

is direct.

Identify each $\beta \in H$ with the permutation of P defined by

$$(x,y)\beta = (x,y\beta)$$

for all $x \in X$, $y \in Y$. Then the action of H on $\Pi G(y)$ is to permute the direct factors; for

$$\beta^{-1}\,\alpha(y)\beta \;=\; \alpha(y\beta)$$

for $\alpha \in G$, $\beta \in H$ and $y \in Y$.

2.1 <u>Definition</u>. The wreath product $W = G \wr H$, of G by H , is the product

$$\Pi G(y) . H \quad .$$

Write $\Pi G(y) = B(G \wr H)$, called the <u>base group</u> of the wreath product, and $H = T(G \wr H)$ the <u>top group</u> . Obviously $G \wr H$ is a splitting extension of $B(G \wr H)$ by $T(G \wr H)$. If $A < G$ then $A(y)$ is the subgroup of $G(y)$ consisting of the elements $\alpha(y)$, $\alpha \in A$. The subgroup $G(y)$, $y \in Y$, is called the <u>yth co-ordinate subgroup</u> of $G \wr H$ and, if Y is finite, the subgroup of all elements

$$\Pi_{y \in Y}\, \alpha(y) \ , \qquad \alpha \in G \ ,$$

is called the <u>diagonal</u> $D(G \wr H)$ of $G \wr H$. (Compare with the terminology of Chapter 1 for direct powers of a group.)

Note that the representation of the bottom group G as a permutation group, does not affect the isomorphism type of the wreath product.

Let K be a group of permutations of a third set Z. Then $(G \wr H) \wr K$ and $G \wr (H \wr K)$ permute elements of the form $((x,y),z)$ and $(x,(y,z))$ respectively, where $x \in X$, $y \in Y$, $z \in Z$. If we identify these by adopting the notation (x,y,z) for both, then the cartesian product $X \times Y \times Z$ of all such ordered triples is the permutand of both wreath products, and it follows easily that in fact

$$(G \wr H) \wr K = G \wr (H \wr K) \quad ,$$

and hence that the expression $G \wr H \wr K$ is unambiguous.

It follows from this that the subgroups $B(G \wr H)$, $T(G \wr H)$, $D(G \wr H)$ may depend on the representation chosen for the group W as a wreath product. It may happen (and will in what follows) that $G \wr H = G_1 \wr H_1$ where $G_1 \not\cong G$ (e.g. $W \wr K = G \wr (H \wr K)$ above); hence the need to specify the representation.

If $K_1, K_2, \ldots, K_n$ are groups not explicitly given as permutation groups, we shall understand

2.2 $$K_1 \wr K_2 \wr \ldots \wr K_n$$

to mean the repeated wreath product of the right regular representations of $K_1, K_2, \ldots, K_n$, and shall call 2.2 the <u>standard wreath product</u> of the K_i in the given order. Usually no distinction will be made between K_i and its right regular representation. We note that the group 2.2 is always transitive but regular if and only if at least $n-1$ of the K_i's are trivial. Also, recall that the only regular, transitive permutation representation of a group is its right regular representation (see M. Hall, Jr. [5], p. 57). (In other contexts (e.g. Peter M. Neumann [18]) the standard wreath product $K_1 \wr K_2$ is called the restricted standard wreath product because the base group is the direct product of the co-ordinate subgroups rather than a cartesian product.)

<u>Lemmas on the wreath product</u>.

Let X , Y , G , H and $W = G \wr H$ be as above. The following few results are required mainly for Chapters 3 and 4, and are for the most part variants on old themes.

2.3 <u>Lemma</u>. If Y is finite and H is transitive then the centralizer of $H = T(G \wr H)$ in $G \wr H$ is the product

$$D(G \wr H) \times Z(H) \quad .$$

The proof is well-known and is therefore omitted (see for example P. Hall [6]).

2.4 Lemma. Provided only that H is a regular permutation group, the centralizer of any element of any co-ordinate subgroup of $G \wr H$ is contained in the base group.

Proof. Let $\alpha(y)$ be an element in the co-ordinate subgroup $G(y)$, $y \in Y$, and let $\gamma\beta$, $e \neq \beta \in H$, $\gamma \in B(G \wr H)$, be any element of $W \setminus B(G \wr H)$. Then

$$(\gamma\beta)^{-1}\alpha(y)\gamma\beta \in G(y)^{\beta} = G(y\beta) .$$

Since H is regular, $y\beta \neq y$ and the proof is complete.

The next two lemmas are slight generalizations of results of Peter M. Neumann [18]. In order to state them concisely it is necessary to define a certain subset $M = M(G \wr H)$ of the base group of $G \wr H$. For brevity write $B = B(G \wr H)$.

Let γ be any element in B : say

$$\gamma = \alpha_1(y_1)\ldots\alpha_n(y_n) ,$$

where the y_i are distinct elements of the permutand Y of H , and $\alpha_i \in G$, $i = 1,\ldots,n$. Consider the set $\Pi(\gamma)$ of all products

$$\Pi\alpha_i = \Pi_{i\in\{1,\ldots,n\}}\alpha_i \in G ,$$

where all possible orders of multiplication are allowed. Obviously if $\Pi\alpha_i \in G'$, the derived group of G , for one order of multiplication then this is so for any order of multiplication.

Write

$$M(G \wr H) = \{\gamma \mid \gamma \in B(G \wr H) \; ; \; \Pi(\gamma) \subseteq G'\} \; .$$

2.5 <u>Lemma</u>. If H is transitive then M is a subgroup of B and

$$M = [H,B] = \operatorname{sgp}\{\alpha^{-1}(y_0)\alpha(y) \mid \alpha \in G \; , \; y_0, y \in Y, \; y_0 \text{ fixed}, \; y \neq y_0\} \; .$$

The proof, which is the same as that of Theorem 4.1 of [18] , is omitted. Although that theorem is proved for the standard wreath product, the proof uses only the transitivity of the representation of the top group. A generalization of Peter M. Neumann's Theorem 4.1 is given in Part II (Theorem 7.4) and the lemma may likewise be deduced from the proof there.

Let η denote the natural homomorphism from $G \wr H$ onto H : that is, for $\beta \in H$ and $\gamma \in B$, set $(\gamma\beta)\eta = \beta$. Then the following lemma tells us that certain normal subgroups of W are in some sense large, and it will also yield some information on the intersections of certain sets of normal subgroups (Corollary 2.7).

2.6 <u>Lemma</u>. If H is transitive and N is a normal subgroup of $G \wr H$ such that $N\eta = H$, then N contains M.

The following proof is a direct generalization of the proof of the corresponding Lemma 8.2 of [18] except where a reference in that proof to other results of [18] is avoided.

Proof of 2.6. By Lemma 2.5, M is generated by the set

$$X = \{\alpha^{-1}(y_0)\alpha(y) \mid \alpha \in G,\ y_0, y \in Y,\ y_0 \text{ fixed, } y \neq y_0\} .$$

It therefore suffices to show that $X \subset N$. This is done as follows. Let $x = \alpha^{-1}(y_0)\alpha(y)$ be any element of X. Since H is transitive there exists an element $\beta \in H$ such that $y_0\beta = y$, and since $N\eta = H$ there exists an element $\gamma\beta \in N$ where $\gamma \in B$, the base group. Let $\mu(y_0)$ be the projection of B onto its direct factor $G(y_0)$ and suppose that $\gamma\mu(y_0) = \alpha_1(y_0)$. Then

$$\alpha_1^{-1}(y_0)\gamma\beta\alpha_1(y_0) = \alpha_1^{-1}(y_0)\gamma\alpha_1(y_0\beta^{-1})\beta = \gamma'\beta \text{ say .}$$

Now $\gamma'\beta \in N$ and, since β^{-1} does not fix y_0, it follows that $\gamma'\mu(y_0) = e$.

Consider the element $\alpha(y_0)$. The commutator $[\gamma'\beta, \alpha(y_0)]$ lies in N, and since γ' commutes with $\alpha(y_0)$ we have

$$[\gamma'\beta, \alpha(y_0)] = [\beta, \alpha(y_0)] .$$

But $[\beta,\alpha(y_0)] = \alpha^{-1}(y_0\beta)\alpha(y_0) = x^{-1}$, and the proof is complete.

This section ends with the following corollary.

2.7 Corollary. If H is transitive and N is a normal subgroup of $G \wr H$ not contained in the base group B , then N contains B' , the derived group of B .

Proof. Let $Y_1 \subseteq Y$ be a set of transitivity for $N\eta$ (or, in other terminology, an orbit relative to $N\eta$) ; that is, $y_1\beta \in Y_1$ for all $\beta \in N\eta$, $y_1 \in Y_1$, and $N\eta$ acts transitively on Y_1 . Write $N\eta = H_1 \leq H$, and consider the wreath product $G \wr H_1$ of G by H_1 where H_1 is of course a permutation group on Y .

Let y_{01} be any fixed element of Y_1 . Then, just as in Lemma 2.6, one can show that the set

$$X_1 = \{\alpha^{-1}(y_{01})\alpha(y_1) \mid \alpha \in G , y_1 \in Y_1 , y_1 \neq y_{01}\}$$

is contained in N . Denote by $H_1|_{Y_1}$ the restriction of H_1 to Y_1 . Then the subgroup

$$\mathrm{sgp}\{G(y_{01}),H_1\}$$

of $G \wr H_1$ is isomorphic to $G \wr H_1|_{Y_1}$ under the isomorphism θ which is the identity on $G(y_{01})$ and maps $\beta_1 \in H_1$ onto its restriction to Y_1 . By Lemma 2.5, $X_1\theta$ generates $M(G \wr H_1|_{Y_1})$, and hence by the definition of the latter subgroup,

$$G'(y_{01}) < N .$$

Now if $G'(y_{01}) < N$, then, by the normality of N and the transitivity of H , $G'(y) < N$ for all $y \in Y$. This gives the desired result since $B'(G \wr H)$ is the direct product of the $G'(y)$.

Direct limits of standard wreath products.

Suppose a sequence $G_1, G_2, \ldots,$ of non-trivial groups is given. From this sequence form a sequence $\{W_n\}$ of standard wreath products by setting

$$2.8 \qquad W_1 = G_1 \; ; \; W_n = W_{n-1} \wr G_n \quad \text{for } n > 1 .$$

We shall be concerned with direct limits arising from a fixed sequence $\{W_n\}$: that is, we shall vary the sequence of embeddings only. Now W_n can be embedded in W_{n+1} in several ways. The first restriction we place on the embeddings is defined in what follows.

Write

$$T_{r,n} = G_{r+1} \wr G_{r+2} \wr \ldots \wr G_n , \quad 1 \leq r < n .$$

Then by 2.8,

$$W_n = W_r \wr T_{r,n}$$

for $1 \leq r < n$. This wreath product is not standard unless $n = 2$. Certain ~~Contain~~ naturally occurring subgroups $B_{r,n}$ of W_n , $1 \leq r \leq n$, are defined as follows.

2.9 <u>Definition</u>. Set

$$B_{r,n} = B(W_r \wr T_{r,n})$$

for $1 \leq r < n$, and

$$B_{n,n} = W_n .$$

Let $\{\theta_n \mid W_n \longrightarrow W_{n+1}\}$ be a sequence of monomorphisms restricted by the conditions

2.10 $$B_{r,n}\theta_n < B_{r,n+1} \quad \text{for } 1 \leq r \leq n .$$

Specific sequences $\{\theta_n\}$ consisting only of embeddings onto either some co-ordinate subgroup or the diagonal (the G_n will be finite), will be chosen at a later stage (Chapters 3, 4). For the rest of this chapter L will denote the direct limit obtained from any fixed sequence $\{\theta_n\}$ satisfying 2.10.

Conditions 2.10 imply that

$$\overline{B}_{n,n} < \overline{B}_{n,n+1} < \cdots .$$

2.11 Definition. For $n \geq 1$ the nth layer of L is the subgroup

$$L_n = \cup_{i=n}^{\infty} \overline{B}_{n,i}$$

It follows from 2.9 and 2.11 that for $n \geq 1$, $L_n < L_{n+1}$, and that

2.12 $$L = \cup_{n=1}^{\infty} L_n .$$

Given two ascending sequences of groups

$$G_1 \leq G_2 \leq \cdots ,$$

and

$$N_1 \leq N_2 \leq \cdots ,$$

for which $N_i \trianglelefteq G_i$ for $i \geq 1$, then $\cup_{i=1}^{\infty} N_i \trianglelefteq \cup_{i=1}^{\infty} G_i$. For, if $g \in \cup_{i=1}^{\infty} N_i$ and $g' \in \cup_{i=1}^{\infty} G_i$, then there exists a k such that $g \in N_k$ and $g' \in G_k$. Since $N_k \trianglelefteq G_k$, $g^{g'} \in N_k$ and the statement is proved.

This, together with Definition 2.11, yields the following lemma.

2.13 Lemma. L_n is normal in L.

The invariance of the layers L_n under isomorphisms between direct limits.

Restrictions are now placed on the given sequence $\{G_n\}$.

Let $G^{(i)}$ denote the ith derived group of a group G ($G^{(o)} = G$; $G^{(1)} = G'$). Then G is said to be soluble of length $s = s(G)$ if $G^{(s)} = E$ but $G^{(s-1)} \neq E$.

For the remainder of this chapter and for Chapter 3 it is assumed that $\{G_n\}$ is a fixed sequence of non-trivial groups satisfying the following condition.

$$2.14 \begin{cases} \text{For all } n \geq 1 \ G_n \text{ is soluble of length } s_n, \\ \text{and } Z(G_n) \text{ contains no elements of order } 2. \end{cases}$$

2.15 Theorem. Under conditions 2.10 and 2.14, for all $n \geq 1$ L_n is the greatest normal subgroup of L which is soluble of length $\Sigma_{i=1}^{n} s_i$.

This theorem is needed mainly in order to prove Theorem 3.1 through the agency of the following corollary.

2.16 Corollary. If 2.14 is satisfied and $\{\theta_n\}$ and $\{\theta_n^*\}$ are two sequences of monomorphisms satisfying 2.10, then for all $n \geq 1$ any isomorphism between the corresponding direct limits L and L^* must map L_n onto L_n^*, where these are the corresponding nth layers.

The proof is immediate from Theorem 2.15.

The proof of 2.15 requires corollaries of the following lemma. Recall that η is the natural homomorphism from $G \wr H$ onto H, where G and H are permutation groups on sets X and Y respectively.

2.17 Lemma. If G and H are soluble permutation groups of soluble lengths $s(G)$ and $s(H)$, and N is a normal subgroup of $G \wr H$ such that $N\eta, \leq H$, has a set of transitivity $Y_1 \subseteq Y$ with $|Y_1| > 2$, then N is soluble of length strictly greater than $s(G)$. In particular if the representation of H is its right regular representation and $|H| > 2$, then the normal closure of H in $G \wr H$ is soluble of length $s(G) + s(H)$.

2.18 Remark. It follows that the standard wreath product of G by H is soluble of length $s(G) + s(H)$. For it is an extension of a group of soluble length $s(G)$ (the base group) by one of soluble length $s(H)$ and $s(G) + s(H)$ is therefore an upper as well as a lower bound for $s(G \wr H)$. This is true without the restriction $|H| > 2$, but this fact will not be needed: it can be proved along lines similar to those of the proof given below. An example to show that the restrictions $|Y_1| > 2$, $|H| > 2$ cannot in general be removed from the statement of the lemma, is provided by the standard wreath product of two 2-cycles. Here the normal closure of the top group is abelian whereas the wreath product itself is soluble of length 2.

Proof of 2.17. Write $N\eta = H_1 \trianglelefteq H$. By the same argument as in the proof of Lemma 2.6, the normality of N in $G \wr H$ and the transitivity of the action of H_1 on Y_1 together imply that the set

$$\{\alpha^{-1}(y)\alpha(y') \mid \alpha \in G;\ y,y' \in Y_1\}$$

is contained in N.

Let y_1, y_2, y_3 be pairwise distinct elements of Y_1. Since H_1 acts transitively on Y_1 there exists $\beta \in H_1$ such that $y_1\beta = y_3$. Then for some $\gamma \in B(G \wr H)$, we have $\gamma\beta \in N$. By the above, the set

$$\{\alpha^{-1}(y_1)\alpha(y_2) \mid \alpha \in G\}$$

is contained in N. From this form the set of commutators

$$\{c_\alpha = [\gamma\beta, \alpha^{-1}(y_1)\alpha(y_2)] \mid \alpha \in G\}\ .$$

Then, expanding the elements of the latter set:

$$\begin{aligned} c_\alpha &= (\gamma^\beta)^{-1}\beta^{-1}\alpha(y_1)\alpha^{-1}(y_2)\beta\gamma^\beta\alpha^{-1}(y_1)\alpha(y_2)\ , \\ &= (\gamma^\beta)^{-1}\alpha(y_3)\alpha^{-1}(y_2\beta)\gamma^\beta\alpha^{-1}(y_1)\alpha(y_2)\ . \end{aligned}$$

Let $\mu(y_3)$ be the projection of $B(G \wr H)$ onto the co-ordinate subgroup $G(y_3)$. Then if $(\gamma^{\beta})\mu(y_3) = \alpha_1(y_3)$, it follows that

$$c_\alpha \mu(y_3) = \alpha^{\alpha_1}(y_3)$$

since y_1, y_2 and $y_2\beta$ are all distinct from y_3. Hence $\mu(y_3)$ maps $\{c_\alpha\}$ onto $G(y_3)$ and the restriction of $\mu(y_3)$ to $\mathrm{sgp}\{c_\alpha\}$ is an epimorphism onto $G(y_3)$. This implies that $\mathrm{sgp}\{c_\alpha\}$ is soluble of length at least $s(G)$. But

$$\mathrm{sgp}\{c_\alpha\} \leq N' ,$$

and the first part of the lemma is proved.

If $|H^{(s-1)}| > 2$ then a simple application of the first part of the lemma proves the second part. Although not necessary for applications, since it is not difficult we indicate the proof of the second part for the case $|H^{(s-1)}| = 2$. In this case the hypothesis $|H| = |H^{(c)}| > 2$ implies that $s(H) \geq 2$ and hence that $|H^{(s-2)}| \geq 6$.

Write $s = s(H)$. It suffices to show that the normal closure of $H^{(s-2)}$ in $G \wr H$ is soluble of length $s(G) + 2$. For, if N is the normal closure of H in $G \wr H$, then $N^{(s-2)}$ is characteristic in N and hence normal in $G \wr H$, and $N^{(s-2)} > H^{(s-2)}$. Then we would have $s(N^{(s-2)}) \geq s(G) + 2$

whence $s(N) \geq s(G) + s(H)$ from which the result follows.

Without risk of confusion the distinction between H and its right regular representation and between corresponding elements, can be dropped. Let T be a transversal for $H^{(s-1)}$ in $H^{(s-2)}$. Then $|T| > 2$. Let τ_1, τ_2, τ_3 be pairwise distinct elements of T and let $\delta \in H^{(s-2)}$ be such that $\tau_1\delta = \tau_3$. Let $e \neq \beta \in H^{(s-1)}$. Then, as above,

$$c_\alpha = [\beta,[\delta,\alpha^{-1}(\tau_1)\alpha(\tau_2)]]$$

lies in the second derived group of the normal closure of $H^{(s-2)}$ in $G \wr H$, for all $\alpha \in G$. Expanding:

$$c_\alpha = \alpha(\tau_1\beta)\alpha^{-1}(\tau_2\beta)\alpha^{-1}(\tau_3\beta)\alpha(\tau_2\delta\beta) \cdot \alpha^{-1}(\tau_1)\alpha(\tau_2)\alpha(\tau_3)\alpha^{-1}(\tau_2\delta) \quad ;$$

and since τ_1, τ_2, τ_3, $\tau_1\beta$, $\tau_2\beta$, $\tau_3\beta$ are pairwise distinct there exists, as before, an epimorphism from $\mathrm{sgp}\{c_\alpha\}$ onto some co-ordinate subgroup of $G \wr H$. But $\mathrm{sgp}\{c_\alpha\} \leq N^{(s)}$ and so $s(N^{(s)}) \geq s(G)$. Thus the lemma is proved.

We now deduce two corollaries of Lemma 2.17 concerning the groups L and L_n of Theorem 2.15.

2.19 <u>Corollary</u>. The nth layer L_n of L, is soluble of length $\Sigma_{i=1}^{n} s_i$.

<u>Proof</u>. Since L_n is the union of an ascending sequence of subgroups isomorphic to direct powers of W_n, the soluble length of L_n is the same as that of W_n. By Remark 2.18 and an easy induction on n, W_n is soluble of length $\Sigma_{i=1}^{n} s_i$ for $n \geq 1$.

2.20 <u>Corollary</u>. For $n \geq 1$ the normal closure in L of any element $\bar{g} \in L$, outside L_n, is soluble of length strictly greater than that of L_n.

<u>Proof</u>. For sufficiently large $m > n$, $\bar{g} \in \bar{W}_m$. Since $\bar{g} \notin L_n$ it follows that $\bar{g} \notin \bar{B}_{n,m}$. Drop the bars and consider

$$g \in W_m \setminus B_{n,m} .$$

The subgroup $B_{n,m}$ is the base group of the wreath product

$$\begin{aligned} W_m &= W_n \wr (G_{n+1} \wr \ldots \wr G_m) \\ &= W_n \wr T_{n,m} , \end{aligned}$$

where $T_{n,m}$ is a transitive permutation group. The hypotheses of Theorem 2.15, together with Lemma 2.17, imply that the normal closure of g in W_m is soluble of length strictly greater than $s(W_n) = s(L_n)$. Thus the proof is complete.

Proof of Theorem 2.15. The theorem is immediate from Lemma 2.13 and Corollaries 2.19 and 2.20.

CHAPTER 3.

THE MAIN THEOREM.

Introduction.

This chapter is taken up with stating and proving the theorem we have been aiming at, and remarking on an implication of it.

Suppose the sequence $\{W_n\}$ of standard wreath products is constructed from the given sequence $\{G_n\}$ of non-trivial groups as in 2.8. Let, for $n \geq 1$, the embedding

$$\delta_n : w_n \longrightarrow \Pi_{x \in G_{n+1}} w_n(x) , \qquad w_n \in W_n ,$$

of W_n onto $D(W_n \wr G_{n+1})$, be called the <u>diagonal embedding</u> of W_n into W_{n+1}. The embedding

$$\gamma_n(x) . w_n \dashrightarrow w_n(x) , \qquad w_n \in W_n ,$$

for some $x \in G_{n+1}$, is to be called the xth <u>co-ordinate embedding</u> of W_n into W_{n+1}.

3.1 <u>Theorem</u>. Suppose that for all n, G_n is soluble of length s_n, has 2-free centre (the conditions 2.14) and is in addition finite. Suppose $\{\theta_n \mid W_n \theta_n < W_{n+1}\}$ and

$\{\theta_n^* \mid W_n\theta_n^* < W_{n+1}\}$ are two sequences of monomorphisms consisting entirely of diagonal and co-ordinate embeddings such that for infinitely many n, one of θ_n, θ_n^* is δ_n, and the other is $\gamma_n(x)$ for some $x \in G_{n+1}$. Then the direct limits L and L^* of $\{W_n\}$, corresponding to $\{\theta_n\}$ and $\{\theta_n^*\}$ respectively, are not isomorphic.

3.2 Remark. Theorem 1.11 now becomes relevant to a certain extent. Clearly $\{\theta_n\}$ and $\{\theta_n^*\}$ satisfy the condition 2.10, and so we may speak of the nth layers L_n and L_n^* of L and L^* respectively (see Definition 2.11). By Corollary 2.16, an isomorphism between L and L^* would map L_n onto L_n^*. Thus it would suffice for proving Theorem 3.1 to prove the latter pair of subgroups non-isomorphic. However this approach cannot work for the following reasons. Consider the three possibilities:

3.2.1 θ_n is the diagonal embedding for only finitely many n;

3.2.2 θ_n is a co-ordinate embedding for only finitely many n;

3.2.3 neither 3.2.1 nor 3.2.2 holds.

By Definition 2.11, for each n, L_n (and L_n^*) is a direct limit of direct powers of W_n. It is not difficult to see that if 3.2.1 holds then $L_n \cong W_n^{(I)}$ for all n (where I is the set of positive integers). If 3.2.2 holds then the hypotheses of Lemma 1.13 are fulfilled and therefore, for each n, L_n is isomorphic to the group L_o of that lemma with W_n replacing the group G occurring there. If 3.2.3 holds then by the proof of Theorem 1.11, we have $L_n \cong L_o^{(I)}$ (with $G = W_n$) .

Thus for each n there are at most three possibilities for L_n (and L_n^*) , corresponding to the situations 3.2.1, 3.2.2 and 3.2.3. In almost all cases the direct limits L and L^* of Theorem 3.1 have isomorphic nth layers for all n .

Preliminary.

It seems appropriate to include the following final lemma in this chapter

3.3 Lemma. Let $G, H_1, \ldots, H_m$, $m \geq 1$, be soluble non-trivial permutation groups with G non-abelian, H_i transitive and $Z(H_i)$ 2-free for $i = 1, \ldots, m$. Form the wreath product

$$W = G \wr H_1 \wr \ldots \wr H_m ,$$

and write $H = H_1 \wr \ldots \wr H_m$, $G_1 = G \wr H_1 \wr \ldots \wr H_{m-1}$. Then $W = G \wr H = G_1 \wr H_m$. Let

$$\theta : W \longrightarrow W^S \quad (S \text{ some index set})$$

be any monomorphism of W into the cartesian power W^S . Then if μ_i is the projection of W^S onto its ith factor $W[i]$, there exists $j \in S$ such that

3.3.1 $$\ker(\theta\mu_j) < B(G \wr H) ;$$

3.3.2 $$H_m\theta\mu_j \not\leq B(G_1 \wr H_m)[j] .$$

<u>Proof</u>. By Lemma 2.17 the normal closure of H_m in $G_1 \wr H_m$ is soluble of length $s(G_1) + s(H_m)$. Thus there exists $j \in S$ such that the normal closure of $H_m\theta\mu_j$ in $W[j]$ is soluble of length $s(G_1) + s(H_m)$. For otherwise the normal closure of H_m would be embeddable in a cartesian product of groups of soluble lengths strictly less that $s(G_1) + s(H_m)$ which is impossible. We shall show that this j satisfies the requirements 3.3.1 and 3.3.2.

By Remark 2.18, $s(W) = s(G) + s(H) = s(G_1) + s(H_m)$. Write $B = B(G \wr H)$ and $B_1 = B(G_1 \wr H_m)$. The factor group W/B' is soluble of length strictly less than W. For,

$$W/B' \cong (G/G') \wr H ,$$

whence by Lemma 2.17 and since G is non-abelian,

$$s(W/B') = s(G/G') + s(H) < s(G) + s(H) = s(W) .$$

Now $W/\ker(\theta\mu_j)$ is soluble of length $s(W)$ since the normal closure of $H_m\theta\mu_j$ in $W[j]$ is embeddable in it. It follows that $\ker\theta\mu_j \not\geq B'$, and this implies, by Corollary 2.7, that $\ker(\theta\mu_j) < B$. Requirement 3.3.1 is therefore satisfied.

Requirement 3.3.2 is satisfied because if $B_1[j]$ were to contain $H_m\theta\mu_j$, it would by its normality in $W[j]$ contain the normal closure of $H_m\theta\mu_j$ in $W[j]$. However this normal closure is soluble of length $s(G_1) + s(H_m)$ whereas $s(B_1[j]) = s(G_1) < s(G_1) + s(H_m)$. This completes the proof of the lemma.

<u>Proof and consequences of the theorem.</u>

We shall now prove Theorem 3.1. In self-explanatory notation, non-starred letters and starred letters represent subgroups or elements of L and L^* respectively.

Assume there exists an isomorphism φ between L and L^*: $L\varphi = L^*$. We shall obtain a contradiction from this assumption.

As the sequences $\{\theta_n\}$ and $\{\theta_n^*\}$ determining L and L^* respectively, differ for infinitely many n, we may assume that for infinitely many n, θ_n is the diagonal embedding of W_n into W_{n+1} and θ_n^* is some co-ordinate embedding of W_n into W_{n+1}. If this is not the case then it will be if the rôles of L and L^* are interchanged.

Now consider $\overline{W}_3$. A set of integers k is defined by the following four conditions:

$$3.4 \quad \begin{cases} (1) & k \geq 3 \; ; \\ (2) & \overline{W}_3\varphi \leq \overline{W}_k^* \; ; \\ (3) & \overline{W}_{k-1} = \overline{D(W_{k-1} \wr G_k)} \; ; \\ (4) & \overline{W}_{k-1}^* = \overline{W_{k-1}(x)}^* \; , \end{cases}$$

where $W_{k-1}(x)$, $x \in G_k$, is a co-ordinate subgroup of $W_{k-1} \wr G_k$.

Since a direct limit is merely the union of an ascending sequence of subgroups, it is clear that any finite set of its elements is contained wholly in some term and then in all succeeding terms of the sequence. Since W_3 is finite, this fact, together with the preceding assumptions on the θ_n , θ_n^* , ensures that there exist infinitely many such integers k . Fix attention on an arbitrary one of them.

Suppose

$$\overline{W}_k\varphi \leq \overline{W}_{k+\ell}^*$$

where $\ell \geq 0$. Again, by the finiteness of W_k , some such ℓ must exist. By Corollary 2.16

$$\overline{W}_k\varphi \leq \overline{B}_{k,k+\ell}^* \; .$$

Now $B_{k,k+\ell}$ is the base group of the (non-standard unless $\ell = 1$) wreath product

$$W_k \wr (G_{k+1} \wr \cdots \wr G_{k+\ell}) \; ,$$

and is therefore a finite direct product of isomorphic copies of W_k . Apply Lemma 3.3 with $W = G \wr H_1 \wr \ldots \wr H_m$ replaced by $\overline{W}_k = \overline{W_2 \wr G_3 \wr \ldots \wr G_k}$ $(m = k - 2)$, W^S replaced by $\overline{B}^*_{k,k+\ell}$ and θ replaced by φ' , the restriction of φ to $\overline{W}_k$. Then, by that lemma, there exists a projection μ of $\overline{B}^*_{k,k+\ell}$ onto a direct factor

$$F^* = \overline{W_k(x)}^*$$

(where $x \in G_{k+1} \wr \ldots \wr G_{k+\ell} = T_{k,k+\ell}$) such that

3.5 $$\ker \varphi'\mu < \overline{B(W_2 \wr (G_3 \wr \ldots \wr G_k))} = \overline{B}_{2,k} ;$$

3.6 $$\overline{G}_k \varphi'\mu \not\leq \overline{B_{k-1,k}(x)}^*$$

From 3.5 it follows that $\overline{W}_3\varphi\mu \neq E$. For if $\overline{W}_3\varphi\mu = E$ then $\overline{W}_3 \leq \ker \varphi'\mu < \overline{B}_{2,k}$. This is not possible because $s(B_{2,k}) = s(W_2) < s(W_3)$. By the way $\overline{W}^*_k$ is embedded in $\overline{W}^*_{k+\ell}$, in fact in $\overline{B}^*_{k,k+\ell}$, every projection of $\overline{W}^*_k$ into a factor $\overline{W_k(x)}^*$ of $\overline{B}^*_{k,k+\ell}$ is either the whole of that factor or the trivial group : that is, $\overline{W}^*_k$ is a part subdirect product of the direct product $\overline{B}^*_{k,k+\ell}$. Since $\overline{W}^*_k\mu \geq \overline{W}_3\varphi\mu \neq E$, we deduce that $\overline{W}^*_k\mu = F^*$: that is μ restricted to $\overline{W}^*_k$ is an isomorphism.

From 3.6 it follows that there exists $g \in \overline{G}_k = \overline{T(W_{k-1} \wr G_k)}$

such that

$$g\varphi\mu \notin \overline{B(W_{k-1} \wr G_k)(x)}^* .$$

Because of the restrictions placed on the embeddings θ_n^*, we have $\overline{B(W_{k-1} \wr G_k)(x)}^* = \overline{B}^*_{k-1,k}\mu$. That is, μ maps the base group of $\overline{W}_k^*$ onto the base group of the direct factor $\overline{W_k(x)}^*$.

Hence

$$g\varphi\mu \notin \overline{B}^*_{k-1,k}\mu \quad .$$

Let $g^* \in \overline{W}_k^*$ be such that $g^*\mu = g\varphi\mu$. Then

$$g^* \notin \overline{B}^*_{k-1,k} \quad .$$

Now, by 3.4 (3) and Lemma 2.3,

$$[\overline{W}_3, g] = E \quad .$$

Applying the mapping $\varphi\mu$:

$$[\overline{W}_3\varphi\mu, g\varphi\mu] = E \quad ;$$

that is

$$[\overline{W}_3\varphi\mu \ , g^*\mu] = E$$

But the restriction of μ to $\overline{W}_k^*$ is an isomorphism, so

$$[\overline{W}_3\varphi, g^*] = E .$$

Thus g^* centralizes $\overline{W}_3\varphi$. But we have from above that $g^* \notin \bar{B}^*_{k-1,k}$, and therefore by Lemma 2.4, $\overline{W}_3\varphi$ intersects any co-ordinate subgroup $\overline{W_{k-1}(x)}^*$ of $\overline{W}^*_k$ trivially. By 3.4(4) this implies that

$$\overline{W}_3\varphi \cap \overline{W}^*_{k-1} = E .$$

Now let k_1 and k_2 be two integers satisfying conditions 3.4 and let $k_2 > k_1$. Then by 3.4 (2),

$$\overline{W}_3\varphi \leq \overline{W}^*_{k_1} .$$

But by the above,

$$\overline{W}_3\varphi \cap \overline{W}^*_{k_2-1} = E ,$$

and since $\overline{W}^*_{k_2-1} \geq \overline{W}^*_{k_1}$, a contradiction has been reached.

The following theorem is an immediate consequence of Theorem 3.1.

3.7 Theorem. The cardinal of the set of non-isomorphic direct limits obtained from the fixed sequence $\{W_n\}$ of groups and all the sequences $\{\theta_n\}$ of embeddings subject to the conditions of Theorem 3.1, is $2^{\aleph_0}$.

Proof. The number of sequences $\{\theta_n\}$ that differ, in the sense of Theorem 3.1, for infinitely many n, is $2^{\aleph_0}$.

3.8 Remarks. Since there are only $2^{\aleph_0}$ possible sequences of embeddings for the sequence $\{W_n\}$, the set of groups we have obtained is large in the sense that it has the largest possible cardinal of any set of groups obtainable as direct limits from the fixed sequence $\{W_n\}$.

The groups W_n are finite and hence the direct limits are all locally finite and countable. Since there are only $2^{\aleph_0}$ isomorphism classes of countable groups, the above theorem provides a method of obtaining large numbers of non-isomorphic, locally finite, countable groups. This will be exploited in Chapter 4.

A further, and at first glance somewhat surprising, property of the direct limits of Theorem 3.1 is now described.

Take a set of $2^{\aleph_0}$ sequences $\{\theta_n\}$ of embeddings satisfying the hypotheses of Theorem 3.1 and also 3.2.3, such that the members of each pair of sequences differ (as in 3.1) for infinitely many n. Denote by $\triangle$ the set of $2^{\aleph_0}$ direct limits of the fixed sequence $\{W_n\}$, obtained from this set of sequences of embeddings. By Theorem 3.1 no two groups in $\triangle$ are isomorphic.

From another point of view each direct limit $L \in \triangle$ is the union of the ascending sequence of layers (see 2.12):

$$L_1 < L_2 < \dots \quad ;$$

Let $\{\theta_n\}$ and $\{\theta_n^*\}$ be any two distinct sequences of embeddings from our original set, and suppose they give rise to L and L^* respectively: $L, L^* \in \Delta$. By Remark 3.2 (which depended on Chapter 1), for all n, the corresponding nth layers L_n and L_n^* are isomorphic (recall that $\{\theta_n\}$, $\{\theta_n^*\}$ satisfy 3.2.3). Further, by Theorem 2.15, any isomorphism between L_n and L_n^* must map L_i onto L_i^* for $i = 1,\ldots,n-1$. Let φ_n and φ_n^* be the identity mappings of L_n and L_n^* respectively. Then for each n, the sequences $\varphi_1,\ldots,\varphi_n$ and $\varphi_1^*,\ldots,\varphi_n^*$ are of the same type. For if α_n is any isomorphism from L_n onto L_n^* and α_i is the restriction of α_n to L_i, $1 \leq i \leq n-1$, then obviously,

$$\varphi_i \alpha_{i+1} = \alpha_i \varphi_i .$$

Thus by Definition 1.3 the above finite sequences are of the same type for all n. However the infinite sequences $\{\varphi_n\}$, $\{\varphi_n^*\}$ are not of the same type: if they were then by Lemma 1.5 we would have $L \cong L^*$.

Thus each of the $2^{\aleph_0}$ distinct groups in Δ is the union of an ascending sequence of groups such that if any two ascending sequences (giving rise to distinct members of Δ) are both terminated at any n, then the finite ascending sequences of groups obtained are indistinguishable (in the sense that the associated finite sequences of embeddings are of the same type).

In other words these ascending sequences of layers provide examples substantiating the following statement:- If $\{H_n\}$ is a given sequence of groups and $\{\theta_n\}$, $\{\theta_n^*\}$ are, as usual, two sequences of monomorphisms, then if the finite sequences $\theta_1,\ldots,\theta_n$ and $\theta_1^*,\ldots,\theta_n^*$ are of the same type for all n, the two direct limits obtained are not necessarily isomorphic.

However we prove the following theorem.

3.9 **Theorem.** Suppose $\{H_n\}$ and $\{H_n^*\}$ are two sequences of groups such that for all n, $H_n \cong H_n^*$ and the automorphism group of H_n, $\mathrm{aut}(H_n)$ say, is finite. If $\{\theta_n \mid H_n\theta_n \leq H_{n+1}\}$ and $\{\theta_n^* \mid H_n^*\theta_n^* \leq H_{n+1}^*\}$ are sequences of embeddings such that for all n, the finite sequences $\theta_1,\ldots,\theta_n$, and $\theta_1^*,\ldots,\theta_n^*$, are of the same type, then $\{\theta_n\}$ and $\{\theta_n^*\}$ are of the same type.

Proof. Without loss of generality we assume that $\{H_n\}$ and $\{H_n^*\}$ are ascending: that is $H_n \leq H_{n+1}$ and $H_n^* \leq H_{n+1}^*$ for all n. For if this is not the state of affairs then we can consider the sequences $\{\overline{H}_n\}$ and $\{\overline{H}_n^*\}$ of subgroups of the respective direct limits. With this assumption the second hypothesis of the theorem is that for all n there exists an isomorphism from H_n onto H_n^* which maps H_i onto H_i^* for

$1 \leq i \leq n$. The same assumption reduces the problem to that of finding a sequence $\{\alpha_n : H_n \longrightarrow H_n^*\}$ of isomorphisms from H_n onto H_n^* such that the restriction of α_{n+1} to H_n, $\alpha_{n+1}|_{H_n}$ say, is α_n, for all n.

For $n \geq 1$, let $I_n(H_1, H_1^*)$ denote the set of all those isomorphisms β_1 from H_1 onto H_1^* that are restrictions of isomorphisms β_n from H_n onto H_n^* for which $H_i\beta_n = H_i^*$ for $1 \leq i \leq n$. That is, $\beta_1 \in I_n(H_1, H_1^*)$ if and only if there exists an isomorphism β_n such that $\beta_n|_{H_1} = \beta_1$ and $H_i\beta_n = H_i^*$ for $1 \leq i \leq n$. The final hypothesis of the theorem is just that $I_n(H_1, H_1^*) \neq \emptyset$ for all n. Clearly,

$$I_n(H_1, H_1^*) \supseteq I_{n+1}(H_1, H_1^*), \quad n \geq 1 .$$

Thus, since $|\mathrm{aut}(H_1)| < \infty$ implies that $|I_n(H_1, H_1^*)| < \infty$ for all n, it follows that

$$\cap_{n=1}^{\infty} I_n(H_1, H_1^*) \neq \emptyset .$$

Define α_1 to be any isomorphism in this intersection and assume inductively that α_i, $1 \leq i \leq k$ has been defined such that for all $n > k$ there exists an isomorphism β_n from H_n onto H_n^* satisfying $\beta_n|_{H_i} = \alpha_i$ for $1 \leq i \leq k$, and $H_i\beta_n = H_i^*$ for $1 \leq i \leq n$. Thus in particular $H_{k+1}\beta_n = H_{k+1}^*$.

Hence if $I_n(H_{k+1}, H^*_{k+1})$ denotes the set of those isomorphisms Ψ_{k+1} from H_{k+1} onto H^*_{k+1} that are

(1) restrictions of isomorphisms Ψ_n from H_n onto H^*_n, for which $H_i\Psi_n = H^*_i$, $1 \leq i \leq n$;

(2) satisfy $\Psi_n|_{H_i} = \alpha_i$, $1 \leq i \leq k$;

then $I_n(H_{k+1}, H^*_{k+1})$ is non-empty and finite for all $n > k$.

As before $I_n(H_{k+1}, H^*_{k+1}) \supseteq I_{n+1}(H_{k+1}, H^*_{k+1})$; hence

$$\cap_{n=k+1}^{\infty} I_n(H_{k+1}, H^*_{k+1}) \neq \phi .$$

Define α_{k+1} to be any element in this intersection. This completes the inductive step. The sequence $\{\alpha_n\}$ is then of the kind required and the proof is complete.

CHAPTER 4.

AN APPLICATION.

Introduction: a theorem.

It is well-known that the property of nilpotency of finite p-groups is not in general possessed by infinite p-groups. In fact there exist infinite p-groups with trivial centre and derived group the whole group. D.H. McLain has constructed such a p-group which has in addition no proper non-trivial characteristic subgroups. (See Kurosh [13], Vol. 2, appendix Q.)

P. Hall [6] has constructed $2^{\aleph_0}$ non-isomorphic countably infinite, locally finite p-groups with the property of verbal completeness defined in Chapter 0 (Definition 0.2). Verbal completeness is obviously much stronger than the requirement that the derived group be the same as the group itself. In fact P. Hall has constructed in the paper [6], for each countable abelian p-group Z, a verbally complete, countably infinite, locally finite p-group whose centre is isomorphic to Z. It follows that there are $2^{\aleph_0}$ non-isomorphic verbally complete p-groups of this sort, since there are this many possibilities for Z.

In the same paper P. Hall leaves open the question of the existence of $2^{\aleph_0}$ distinct p-groups of this kind, with trivial centres. With the help of a theorem of his from [6], a theorem of Peter M. Neumann [18] and some results of the previous chapters, this question is answered here in the affirmative.

4.1 Theorem. For every prime p there exist $2^{\aleph_0}$ non-isomorphic, verbally complete, countably infinite, locally finite p-groups with trivial centres.

Proof of the theorem.

A sufficient condition for verbal completeness of a countably infinite, locally finite group is given by Theorem 3 of [6]. For the purposes of this chapter it is convenient to formulate it as follows.

4.2 Theorem. Suppose a countably infinite, locally finite p-group P (p prime) is the direct limit of a sequence $\{P_n \mid n = 1,2,\ldots\}$ of finite p-groups, with monomorphisms $\{\varphi_n \mid P_n\varphi_n \leq P_{n+1}\}$. Then P is verbally complete if, for infinitely many n, φ_n is of the same embedding type as the embedding of the diagonal $D(P_n \wr C_p)$ as a subgroup of the standard wreath product $P_n \wr C_p$, where C_p is a p-cycle.

By setting, in Theorem 3.1, $G_n \cong C_p$ for all $n \geq 1$, with p a prime > 2, Theorem 3.7 is made to yield immediately $2^{\aleph_0}$ non-isomorphic, countably infinite, locally finite p-groups. Moreover, by an easy application of Lemma 2.3, all but one have trivial centre, the exception being that obtained by taking θ_n to be an embedding onto the diagonal of $W_{n+1}, = W_n \wr C_p$, for all $n \geq n_0$, for some positive integer n_0. Of the remaining ones, all but one satisfy the hypotheses of Theorem 4.2, the exception being that one obtained by taking θ_n to be an embedding of W_n onto some co-ordinate subgroup of W_{n+1} for all $n \geq n_1$, for some n_1. Thus Theorem 4.1 is proved already for $p > 2$.

However for $G_n \cong C_p$, $p \geq 2$, simple proofs of results corresponding to the key lemmas of Chapters 2 and 3 are possible.. We prove these results now in order to get Theorem 4.1 for $p = 2$. Although the proofs are valid for arbitrary prime p, for simplicity they are proved only for the prime 2.

4.3 Lemma. (Cf. Corollary 2.16.) Suppose $G_n \cong C_2$, a 2-cycle, for all $n \geq 1$ and form the sequence $\{W_n\}$ of standard wreath products as defined by 2.8. Let $\{\theta_n\}$ and $\{\theta_n^*\}$ be two sequences of monomorphisms giving rise to direct limits L and L^* respectively, and satisfying 2.10 so that the layers L_n and L_n^* are defined for all n. Then for $n \geq 2$, any isomorphism φ from L onto L^* must map L_n onto L_n^*.

Before this can be proved the following fact must be established.

4.4 Lemma. If $n > 2$, the normal closure in W_n of any element outside the base group $B_{n-1,n} = B(W_{n-1} \wr G_n)$, has centre a 2-cycle.

Proof. Let $k\beta \in W_n \setminus B_{n-1,n}$ where β generates $G_n \cong C_2$ and $k \in B_{n-1,n}$. Write N for the normal closure of $k\beta$ in W_n. By Lemmas 2.5 and 2.6, N contains

$$M = \mathrm{sgp}\{\alpha^{-1}(e)\alpha(\beta) \mid \alpha \in W_{n-1}\} .$$

Since the projection of M on each co-ordinate subgroup of $W_{n-1} \wr G_n$ is evidently the whole of that co-ordinate subgroup, $Z(M)$ is contained in

$$Z(B_{n-1,n}) = Z(W_{n-1}(e)) \times Z(W_{n-1}(\beta)) ,$$

the centre of $B_{n-1,n}$. Since $n > 2$, W_{n-1} is non-abelian and so therefore is $B_{n-1,n}$. By Corollary 2.7, $N > B'_{n-1,n}$, and so certainly $N > W'_{n-1}(e) > E$. Now by Lemma 2.4, the centralizer of $W'_{n-1}(e)$ in $W_{n-1} \wr G_n$ lies in the base group. Hence $Z(N)$ lies in $B_{n-1,n}$. Now k centralizes $Z(B_{n-1,n})$, and so $Z(N)$ centralizes β and hence G_n. Thus by Lemma 2.3 the required centre $Z(N)$ is

$$Z(B_{n-1,n}) \cap D(W_{n-1} \wr G_n) ,$$

which is clearly cyclic of order 2 .

Note that, since $n > 2$, we have

4.5 $$|M| > |W_{n-1}| \ .$$

For, if we write $X = \{\alpha^{-1}(e)\alpha(\beta) \mid \alpha \in W_{n-1}\}$, then $|X| = |W_{n-1}|$. But M , $= sgp(X)$, contains $W'_{n-1}(e)$ by Lemma 2.5. Since $n > 2$, W_{n-1} is non-abelian and so $W'_{n-1}(e) \not\subseteq X$. It follows that 4.5 is true. This will be used in the following proof.

Proof of Lemma 4.3. As the beginning of a reductio ad absurdum, suppose there is an element $\bar{g} \in L_n$ such that $\bar{g}\varphi \notin L_n^*$ (for some fixed $n \geq 2$). Then there exists $m > n$ such that

$$\bar{g}\varphi \in L_m^* \setminus L_{m-1}^* \ ,$$

since L^* is the union of the layers L_i^* . Now for all $i \geq 1$, L_i^* is the union of the following ascending sequence of groups isomorphic to finite direct powers of W_i :

$$\bar{W}_i^* < \bar{B}_{i,i+1}^* < \bar{B}_{i,i+2}^* < \ldots \ .$$

Therefore, for some $\ell \geq m$,

$$\bar{g}\varphi \in \bar{B}^*_{m,\ell} \setminus \bar{B}^*_{m-1,\ell} \text{ .}$$

Write $\bar{g}\varphi = \bar{h}^*$. Then, omitting the bars and stars, consider

$$h \in B_{m,\ell} \setminus B_{m-1,\ell} \text{ .}$$

For the projection μ say, of $B_{m,\ell}$ onto at least one direct factor $W_m(x)$ say, where $x \in T_{m,\ell}$, we must have

$$h\mu \notin B(W_{m-1} \wr G_m)(x) = B_{m-1,m}(x) \text{ ,}$$

the base group of $W_m(x)$. Let K denote the normal closure of h in $B_{m,\ell}$. Since $h\mu$ does not lie in the base group of $W_m(x)$, it follows from the proof of Lemma 2.6 that

$$[h', W_m(x)] \geq M(x) \text{ ;}$$

here $M(x)$ is the subgroup generated by all elements $(\alpha^{-1}(e)\alpha(\beta))(x)$, $\alpha \in W_{m-1}$, where β is the generator of $G_m \cong C_2$; and h' is some conjugate of h by an element in $W_m(x)$. Thus $K > M(x)$ and clearly $K\mu > M(x)$.

Applying Lemma 4.4 and the inequality 4.5 to the normal closure $K\mu$ of $h\mu$ in $W_m(x)$, we see that $K\mu$ has centre a 2-cycle and $|M(x)| > |W_{m-1}|$.

By Lemma 2.13, $L_n \lhd L$, $L_n^* \lhd L^*$. Therefore $\overline{K}^*\varphi^{-1} < L_n$. But K is finite and so there exists $s \geq 1$ such that

$$\overline{K}^*\varphi^{-1} \leq \overline{B}_{n,n+s} \quad .$$

The subgroup $\overline{B}_{n,n+s}$ is isomorphic to a (finite) direct power of W_n . It follows that K is embeddable in a finite direct power of W_n. By Lemma 0.1, there exists a finite set $\{N_i \mid i = 1,\ldots,t\}$ of normal subgroups of K such that $\cap_{i=1}^{t} N_i = E$ and K/N_i is embeddable in W_n for $i = 1,\ldots,t$. Write $M_i = N_i \cap M(x)$. Then $\cap_{i=1}^{t} M_i = E$ and $M_i \lhd K\mu$, $i = 1,\ldots,t$. Since $K\mu$ is a finite 2-group with cyclic centre, it follows that at least one M_i is trivial. It follows that for this i , K/N_i contains a subgroup isomorphic to $M(x)$. But K/N_i is embeddable in W_n and $|W_n| \leq |W_{m-1}| < |M(x)|$. A contradiction has been reached and hence $L_n\varphi \leq L_n^*$. Similarly $L_n^*\varphi^{-1} \leq L_n$, and the proof is complete.

4.6 Lemma. (Cf. Lemma 3.3.) Suppose θ is an embedding of $W_n = W_{n-1} \wr G_n$, $n > 2$, into W_n^S (S some index set). Then if μ_i , $i \in S$, is the projection of W_n^S onto its ith factor, there exists at least one element $j \in S$ such that

4.6.1 $$\ker(\theta\mu_j) = E \quad ;$$

4.6.2 $$G_n\theta\mu_j \not\leq B(W_{n-1} \wr G_n)[j] \; .$$

<u>Proof</u>. Since the centre of the 2-group W_n is cyclic, as in the proof of Lemma 4.4, the family of normal subgroups (the kernels of the $\theta\mu_i$, $i \in S$) associated with the embedding θ , must contain at least one trivial member. Choose j so that $\ker(\theta\mu_j) = E$; this j satisfies 4.6.1. Note that then $\theta\mu_j$ is an isomorphism.

Then 4.6.2 is immediate from the following theorem of Peter M. Neumann [18].

4.7 <u>Theorem</u>. Given the standard wreath product $G \wr H$ of two arbitrary groups, then the base group is characteristic except when H is a 2-cycle and G is the splitting extension of a 2-free abelian group, containing square roots of all its elements, by a 2-cycle which transforms each element into its inverse.

Now the reason for taking $n > 2$ in the hypotheses of Lemma 4.6, emerges. For the base group of $C_2 \wr C_2$ is not characteristic since $W_1 = C_2$ is of the form of the G of the theorem. But for $n \geq 2$, W_n is not of that form. The proof of Lemma 4.6 is now complete.

From these lemmas we deduce the following theorem.

4.8 <u>Theorem</u>. (Cf. Theorem 3.1.) With $G_n \cong C_2$ for all $n \geq 1$, and otherwise the same hypotheses as in Theorem 3.1, the direct limits L and L^* are not isomorphic.

<u>Proof</u>. Exactly the same proof as that of 3.1 now works for 4.8, invoking Lemma 4.3 in place of Corollary 2.16 and Lemma 4.6 in place of Lemma 3.3.

<u>Proof of Theorem 4.1</u>. Theorem 4.8 gives the result by the same argument as that following Theorem 4.2.

PART II

GROUPS GENERATING CROSS PRODUCT VARIETIES

CHAPTER 5.

AN UPPER BOUND FOR $\ell(\underline{\underline{N}}\,\underline{\underline{A}}_n)$.

Introduction.

As stated in Chapter 0, $\underline{\underline{N}}\,\underline{\underline{A}}_n$ denotes the product variety of any fixed variety $\underline{\underline{N}}$ nilpotent of class c and of exponent m, by $\underline{\underline{A}}_n$ the variety of all abelian groups of exponent n, where $(m,n) = 1$. The main result of Part II, from or for the proof of which the other results of independent interest are obtained, is now formulated as a theorem.

5.1 Theorem. Provided $c > 1$ the variety $\underline{\underline{N}}\,\underline{\underline{A}}_n$ is generated by its c-generator groups but not by its $(c-1)$-generator groups. That is $\ell(\underline{\underline{N}}\,\underline{\underline{A}}_n) = c$ if $c > 1$. For $c = 1$, $\ell(\underline{\underline{N}}\,\underline{\underline{A}}_n) = 2$.

In this chapter the following half of Theorem 5.1 is proved.

5.2 For $c > 1$, $\ell(\underline{\underline{N}}\,\underline{\underline{A}}_n) \leq c$ and for $c = 1$, $\ell(\underline{\underline{N}}\,\underline{\underline{A}}_n) = 2$.

Proof of 5.2.

Since a locally finite variety is generated by its critical groups, the following lemma, part of which is due to L.G. Kovács, is a first step towards a proof of 5.2.

5.3 Lemma. For any critical group G in $\underline{\underline{N}}\,\underline{\underline{A}}_n$, $d(G) \leq c + 1$. This is true for all $c \geq 1$.

(The symbol $d(G)$ denotes the least number of generators generating G.)

<u>Remark</u>. This bound is certainly attained by some critical groups for $c = 1$ and all suitable pairs m, n. In Chapter 6 we shall see that the bound is attained for $c = 2$ at least for certain exponents m and n.

<u>Proof of 5.3</u>. Firstly, for all $c \geq 1$ we obtain the bound $2c$ for $d(G)$. The following argument is due to L.G. Kovács and is derived in part from an argument in the paper [10] of Kovács and Newman.

Lemma 2.4.2 of Oates and Powell [20] is used:

5.4 If a group G has a set of normal subgroups $M_1, \ldots, M_s$ and a subgroup L such that

5.4.1 $G = LM_1 \ldots M_s$;

5.4.2 G is not generated by L together with any proper subset of the set $\{M_1, \ldots, M_s\}$;

5.4.3 $\left[M_{\pi(1)}, \ldots, M_{\pi(s)}\right] = E$ for every permutation π of the integers $1, \ldots, s$;

then G is not critical.

Now return to the critical group G in $\underline{\underline{N}}\,\underline{\underline{A}}_n$. We may assume G is non-abelian since otherwise G is cyclic and $d(G) = 1$. Since any critical group is finitely generated and $\underline{\underline{N}}\,\underline{\underline{A}}_n$ is locally finite, G is finite. Let $F = F(G)$ be the greatest normal nilpotent subgroup (the Fitting subgroup) and $\Phi = \Phi(G)$ the Frattini subgroup of G. Since Φ is nilpotent and normal, $\Phi \leq F$. F is a p-group for some prime p otherwise G would not be monolithic (i.e. it would not possess a unique minimal normal subgroup). Furthermore $p|m$ where m is the exponent of $\underline{\underline{N}}$. For, by the definition of a product variety and since G has been assumed non-abelian, G is an extension of a non-trivial subgroup S in $\underline{\underline{N}}$ by a group in $\underline{\underline{A}}_n$ and we must have $S \leq F$. Since $(m,n) = 1$ there is, by the Schur-Zassenhaus Theorem, a complement L of F in G . It follows that $L\Phi/\Phi$ $(\cong L)$ is a complement of F/Φ in G/Φ . By Theorems 2, 5 and 9 of Gaschütz [4] , F/Φ is an elementary abelian p-group. (Clearly $\Phi \neq F$.) Write

$$F/\Phi = M_1/\Phi \times \ldots \times M_s/\Phi ,$$

where M_i/Φ, $i = 1,\ldots,s$, is an elementary abelian minimal normal subgroup of G/Φ . That such a decomposition exists is a consequence of Maschke's Theorem; F/Φ may be regarded as an L/Φ-module over $GF(p)$. Then $G = LM_1\ldots M_s$ and conditions 5.4.1 and 5.4.2 are satisfied. If $s > c$, condition 5.4.3 would also be satisfied, contrary to the criticality of G . Thus $s \leq c$.

Since Φ is the finite set of non-generators of G , it follows that $d(G/\Phi) = d(G)$. We can therefore restrict our attention to G/Φ . Write $G/\Phi = G_1$, $L\Phi/\Phi = L_1$, $F/\Phi = F_1$

and $M_i/\Phi = N_i$, $i = 1,\dots,s$. Let x_i be any non-trivial element from N_i , $i = 1,\dots,s$. Then

$$G_1 = \mathrm{sgp}\{L_1, x_1, \dots, x_s\} , \tag{5.5}$$

since the conjugates of x_i under L_1 must together generate the whole of the minimal normal subgroup N_i . We shall now bound $d(L_1)$. Let K_i be the kernel of the representation of L_1 on N_i , $i = 1,\dots,s$: that is $K_i = C_{L_1}(N_i) \lhd L_1$. By Theorem 10 of Gaschutz [4] , $F(G/\Phi) = F/\Phi$. This, together with the abelianness of L_1 , implies that $C_{G_1}(F_1) \leq F_1$, for otherwise there would be a normal nilpotent subgroup of G_1 properly containing F_1 . Hence L_1 is faithfully represented on $F_1 = N_1 \times \dots \times N_s$ and so $\cap_{i=1}^{s} K_i = E$. Now L_1/K_i is abelian and is represented faithfully and irreducibly on N_i . By a classical theorem of representation theory this implies that L_1/K_i is cyclic. Thus L_1 contains s normal subgroups intersecting trivially and with cyclic factor groups. It follows that L_1 is embeddable in the direct product $L_1/K_1 \times \dots \times L_1/K_s$ of s cycles and that $d(L_1) \leq s$. We have then from 5.5 that

$$d(G) = d(G_1) \leq 2s \leq 2c .$$

Secondly, this bound is reduced to $c + 1$ with the help of the following lemma.

5.6 Lemma. Let B be a finite abelian group with $\leq s$ generators and let $B_1,\ldots,B_{s-1}$ be $s-1$ subgroups such that B/B_i is cyclic, $i = 1,\ldots,s-1$. Then there exists a set $\{g_1,\ldots,g_s\}$ of generators of B such that $g_i \in B_i$ for $i = 1,\ldots,s-1$.

Proof. If B is trivial then so also is the lemma. Assume $B \neq E$.

Write B as the direct product of its Sylow subgroups: $B = S_{p_1} \times \cdots \times S_{p_k}$ say. Then, for each j, $1 \leq j \leq k$, S_{p_j} and $B_1 \cap S_{p_j},\ldots,B_{s-1} \cap S_{p_j}$ satisfy the hypotheses of the lemma. If there exists a set $\{g_{1p_j},\ldots,g_{sp_j}\}$ of generators of S_{p_j} such that $g_{ip_j} \in B_i \cap S_{p_j}$, $i = 1,\ldots,s-1$, put

$$g_i = \Pi_{j=1}^{k} g_{ip_j}, \quad i = 1,\ldots,s.$$

Then $\{g_1,\ldots,g_s\}$ satisfies the requirements of the lemma.

Hence it suffices to prove the lemma for B a p-group. With this assumption the lemma is proved by induction on s. For $s = 1$ the lemma holds vacuously. Suppose $s > 1$ and assume it true for $s-1$. Write $B = C_{p^{n_1}} \times \ldots \times C_{p^{n_s}}$ where $n_i \geq 0$, $i = 1,\ldots,s$ and $n_j \leq n_s$ for $j < s$, and let y_i generate $C_{p^{n_i}}$.

Consider the case $B_1 < B$. In this case there exists some element $y = y_1^{m_1} \dots y_{s-1}^{m_{s-1}} y_s \notin B_1$ since elements of this form generate B. Hence $B = C_{p^{n_1}} \times \dots \times C_{p^{n_{s-1}}} \times \mathrm{sgp}\{y\} = A \times \mathrm{sgp}\{y\}$ say. For this decomposition of B, the projection of B_1 on A must be the whole of A since, if $a \neq e$ were not in this projection then, modulo B_1, the set $\{a,y\}$ would generate a non-cyclic group.

Thus for $B_1 \leq B$ there is a generating set $\{c_1,\dots,c_{s-1},g_s\}$ for B such that $c_1,\dots,c_{s-1} \in B_1$.

Write $H = \mathrm{sgp}\{c_1,\dots,c_{s-1}\} \leq B_1$ and consider the cyclic group $HB_i/B_i \cong H/H\cap B_i$, $i = 2,\dots,s-1$. The groups H and $H \cap B_2,\dots,H \cap B_{s-1}$ satisfy the conditions of the lemma and so, by the inductive hypothesis there exist generators $g_1,\dots,g_{s-1}$ for H such that $g_i \in H \cap B_i \leq B_i$, for $i = 2,\dots,s-1$. Since $g_1 \in B_1$, the set $\{g_1,\dots,g_s\}$ satisfies the requirements of the lemma. This completes the proof of the inductive step and the proof of the lemma.

For the application of this to the proof of Lemma 5.3 we return to 5.5 : $G_1 = \mathrm{sgp}\{L_1,x_1,\dots,x_s\}$. The groups L_1 and $K_1,\dots,K_{s-1}$ satisfy the hypotheses of 5.6 and hence there is a generating set $\{\ell_1,\dots,\ell_s\}$ for L_1 with $\ell_i \in K_i$ for $i = 1,\dots,s-1$. Since K_i centralizes x_i and they have coprime orders, the set $\{\ell_1x_1,\dots,\ell_{s-1}x_{s-1}, \ell_s,x_s\}$ generates G_1. This completes the proof of 5.3.

One further lemma will complete the proof of 5.2. Since

$\underline{\underline{N}}\,\underline{\underline{A}}_n$ is generated by its critical groups we have immediately from 5.3 that $\ell(\underline{\underline{N}}\,\underline{\underline{A}}_n) \leq c + 1$. As remarked above, in Chapter 6, for $c = 2$ and certain m , n , critical groups in $\underline{\underline{N}}\,\underline{\underline{A}}_n$ having not fewer than 3 generators are constructed. Thus we cannot hope to obtain the upper bound c for $\ell(\underline{\underline{N}}\,\underline{\underline{A}}_n)$ by considering the critical groups alone. However L.G. Kovács has proved the following result.

5.7 Lemma. For each critical group G in $\underline{\underline{N}}\,\underline{\underline{A}}_n$ with $s > 1$ defined as previously, there exists an s-generator permutational verbal wreath product lying in $\underline{\underline{N}}\,\underline{\underline{A}}_n$ and having G as a factor.

Proof. We retain the notation of the proof of Lemma 5.3 for the relevant subgroups etc. of the critical group G . Identify L_1 with L under the mapping $\ell\Phi \longrightarrow \ell$, $\ell \in L$. Then L/K_i , $i = 1,\ldots,s$, is a cyclic group of order dividing n . Let Z_i be a group isomorphic to L/K_i , $i = 1,\ldots,s$, such that $Z_j \cap Z_k = \emptyset$ for $j \neq k$. Let $\theta_i : L/K_i \longrightarrow Z_i$ be an isomorphism, $i = 1,\ldots,s$. Form the set-theoretical union $\bigcup_{i=1}^{s} Z_i$, $= Z$ say. Then for each $z_i \in Z_i$, $i = 1,\ldots,s$, a permutation ζ_i of Z is defined as follows:

$$z\zeta_i = zz_i \quad \text{if} \quad z \in Z_i \ ;$$

$$z\zeta_i = z \quad \text{if} \quad z \notin Z_i \ .$$

The group Q generated by all such permutations is isomorphic to $Z_1 \times \ldots \times Z_s$ and the restriction to $Z_i \subseteq Z$ of this group of permutations is the right regular representation of Z_i. Take $|Z|$ distinct isomorphic copies of the p^α-cycle C_{p^α} where p^α is the exponent of $F(G)$, and denote them by $C_{p^\alpha}(z)$, $z \in Z$. Form the verbal $\underline{\underline{N}}$-product

$$K = \Pi^{\underline{\underline{N}}}_{z \in Z} C_{p^\alpha}(z) \quad ,$$

as defined in Chapter 0, and split-extend K by Q in the usual way for permutational wreath products: that is, the action of $\zeta \in Q$ on $C_{p^\alpha}(z)$ is defined by

$$(a(z))^\zeta = a(z\zeta) \quad ,$$

for $a \in C_{p^\alpha}$. Then KQ is the permutational verbal wreath product mentioned in the lemma. Obviously $KQ \in \underline{\underline{N}}\,\underline{\underline{A}}_n$.

We shall now choose a subgroup of KQ and find an epimorphism from this subgroup onto G. Let Ψ be the isomorphism from $Z_1 \times \ldots \times Z_s$ onto Q defined by

$$z_i\Psi = \zeta_i \quad , \quad z_i \in Z_i \quad , \quad i = 1,\ldots,s \quad .$$

Then the mapping

$$\Theta : \ell \longrightarrow (\ell K_1)\theta_1\Psi.(\ell K_2)\theta_2\Psi \ldots (\ell K_s)\theta_s\Psi \; ; \qquad \ell \in L \, ,$$

is a monomorphism of L into Q. Write $L\theta = L_2 \leq Q$. Let $b_i \in M_i \setminus \Phi(G)$, e_i be the identity of Z_i, $i = 1,\ldots,s$, and let a generate C_{p^α}. Then the mapping

$$a(e_i) \longrightarrow b_i \quad , \quad i = 1,\ldots,s \; ;$$

$$\ell\theta \longrightarrow \ell \quad , \quad \ell \in L \; ,$$

can be extended to an epimorphism of KL_2 onto G. This follows from the structure of G obtained in the proof of Lemma 5.3, from the freeness of K in $\underline{\underline{N}}$ and the well-known von Dyck's Theorem (Kurosh [13], vol. 1, p. 130).

Finally we calculate the smallest number of generators needed to generate KQ; i.e. $d(KQ)$. If $s = 1$, KQ is a 2-generator group. If $s > 1$, KQ is an s-generator group: for, if $a(e_i)$ generates $C_{p^\alpha}(e_i)$ and z_i generates Z_i, $i = 1,\ldots,s$, then KQ is generated by the set $\{a(e_i), z_i\Psi \mid i = 1,\ldots,s\}$; but $a(e_j)$ and $z_k\Psi$ commute if $j \neq k$ and have coprime orders and hence

$$\{a(e_i).(z_{(i+1) \bmod s}\Psi) \mid i = 1,\ldots,s\}$$

also generates KQ. This completes the proof of 5.7 and thence 5.2.

CHAPTER 6.

CRITICAL GROUPS IN $\underline{\underline{N}}\,\underline{\underline{A}}_n$

Introduction.

As noted in Chapter 0, the theorem of Hanna Neumann proved below (Theorem 6.2) gives rise to the following question.

6.1 Does there exist a variety generated by its k-generator groups and also by a set S of critical groups some or all of which require more than k generators ?

To put the question differently we ask whether or not there exists a variety generated by its k-generator groups and also by its critical groups, yet containing a critical group requiring $> k$ generators. This is answered below affirmatively for $k = 2$. The relevant variety is $\underline{\underline{N}}\,\underline{\underline{A}}_q$ where $c = 2$, $m = p$ and p and q are primes such that $q|p - 1$. (Note that there is only one variety of class 2 and exponent p $(p > 2)$ since any factor group of a free group of such a variety by a non-trivial verbal subgroup, is abelian.) By 5.2 this variety is generated by its 2-generator groups.

However it turns out that if the single strictly 3-generator critical group in $\underline{\underline{N}}\,\underline{\underline{A}}_q$ is contained in a subvariety $\underline{\underline{V}}$ of $\underline{\underline{N}}\,\underline{\underline{A}}_q$ such that $\ell(\underline{\underline{V}}) = 2$, then it can be omitted from any set of critical groups generating $\underline{\underline{V}}$. Thus if we insist that in the question 6.1 not all of the critical groups of S requiring $>k$ generators be omittable in this sense, it remains completely unanswered.

All the critical groups of $\underline{\underline{N}}\,\underline{\underline{A}}_q$, with $\underline{\underline{N}}$ as above, are found and from them the structure of the complete lattice of subvarieties of $\underline{\underline{N}}\,\underline{\underline{A}}_q$ is determined.

Sets of critical groups.

There follow three results connecting properties of a set S of critical groups with $\ell(\mathrm{var}(S))$. The following theorem is due to Hanna Neumann.

6.2 Theorem. The variety generated by a single critical group G , with $d(G) = k$, is not generated by its $(k - 1)$-generator groups.

Proof. Suppose the theorem false and that G is a counterexample, G critical, $d(G) = k$. Denote by F_{k-1} the reduced free group of rank $k - 1$ of $\mathrm{var}(G)$. Then $\mathrm{var}(G) = \mathrm{var}(F_{k-1})$ by the supposition. Hence F_{k-1} is isomorphic to a factor of a cartesian power G^I where $I \neq \emptyset$ is some index set. Since F_{k-1} is free in $\mathrm{var}(G)$, it is in fact embeddable in G^I : $F_{k-1} \cong A \leq G^I$, say. Let θ_i , $i \in I$, be the projection of G^I on its ith co-ordinate. If $A\theta_i = G$ for some $i \in I$, G would be isomorphic to a factor group of $A \cong F_{k-1}$ and would then have fewer than k generators. Thus $A\theta_i < G$ for all $i \in I$. Hence F_{k-1} is in the variety generated by all $A\theta_i$ which is in turn in the variety generated by the proper subgroups of G. This gives us a contradiction and completes the proof.

The next lemma uses a technique contained in the paper [11] of Kovács and Newman.

6.3 Lemma. If a Cross variety $\underline{\underline{V}}$ (that is a variety generated by a single finite group) is generated by its k-generator groups, then it is generated by its k-generator critical groups.

Proof. Let F_k be the free group of rank k of $\underline{\underline{V}}$. Then F_k is embeddable in a finite direct product of critical groups in $\underline{\underline{V}}$ since $\underline{\underline{V}}$ is generated by its critical groups and F_k is finite. We choose, as in the paper quoted above, a "minimal representation" for F_k as a subgroup of such a direct product. The process is as follows. Let $S_1 = \{G_1, \ldots, G_t\}$ be a finite set of pairwise non-isomorphic critical groups in $\underline{\underline{V}}$ such that F_k can be embedded in a finite direct product D_1 of isomorphic copies of them by means of a monomorphism θ_1. A second set S_2 of critical groups is formed from S_1 in the following way. Let T_1 be the set of those $G_i \in S_1$ such that the projections of $F_k\theta_1$ into all the direct factors of D_1 isomorphic to G_i, are properly contained in these direct factors. Let R_1 be the set of proper critical factors of the groups in T_1. Then S_2 is to contain precisely one representative from each isomorphism class represented in the set $(S_1 \setminus T_1) \cup R_1$. Evidently S_2 generates $\underline{\underline{V}}$.

This procedure is repeated with S_2 in place of S_1 to obtain S_3 and so on. After a finite number ℓ of steps, $T_{\ell+1}$ will be empty since all the groups considered are finite. Thus S_ℓ is a set of critical groups generating $\underline{\underline{V}}$ all of which are homomorphic images of F_k. They are therefore all k-generator and the lemma is proved.

The following rather special corollary is applied in the next section.

6.4 Corollary. Let $\underline{\underline{V}}$ be a Cross variety generated by its k-generator groups. If $G \in \underline{\underline{V}}$ is a critical group with $d(G) > k$ such that G can be embedded in some free group of $\underline{\underline{V}}$, then G can be embedded in some k-generator critical group in $\underline{\underline{V}}$.

Proof. By Lemma 6.3 $\underline{\underline{V}}$ is generated by its k-generator critical groups. Suppose G can be embedded in F_ℓ the free group of rank ℓ of $\underline{\underline{V}}$. The freeness of F_ℓ implies that it can be embedded in a (finite) direct product D of k-generator critical groups. Thus G can be likewise embedded: say $G \cong A \leq D$. The projections of A into the critical direct factors of D are all isomorphic to factor groups of G and the variety they generate contains G. Therefore, by the criticality of G at least one such projection must be isomorphic to G and the corollary is proved.

<u>The critical groups of</u> $\underline{\underline{N}}\,\underline{\underline{A}}_q$.

For the rest of this chapter only, $\underline{\underline{N}}$ will be assumed to be nilpotent of class 2 and of exponent p where $q|p-1$ and the primes p and q are fixed. By 5.2 $\underline{\underline{N}}\,\underline{\underline{A}}_q$ is generated by its 2-generator groups. We shall now construct all the critical groups in $\underline{\underline{N}}\,\underline{\underline{A}}_q$ (and so incidentally the promised strictly 3-generator critical group) and thence draw its subvariety lattice.

There follows a résumé of the properties of critical groups in $\underline{\underline{N}}\,\underline{\underline{A}}_q$ which were elicited for the more general case in the proof of Lemma 5.3.

If G is critical, $G \in \underline{\underline{N}}\,\underline{\underline{A}}_q$, then G is a splitting extension of a p-group F by a subgroup L of $C_q \times C_q$. Further, if F is non-trivial, then $F/\Phi(G) = N_1 \times N_2$ where N_i , $i = 1,2$, is either trivial (but not both N_1 and N_2 are trivial) or an elementary abelian minimal normal subgroup of $G/\Phi(G)$. If N_i , $i = 1,2$, is non-trivial it can also be regarded as an irreducible L-module over GF(p) . From this point of view L is represented faithfully on $N_1 \times N_2$ and if K_i , $i = 1,2$, is the kernel of the representation of L on N_i , then L/K_i is a q-cycle or else trivial.

We shall now examine the possibilities for the modules N_1 and N_2 . The condition $q|p-1$ simplifies the further structure of G considerably. In general if p and q are arbitrary distinct primes, the irreducible C_q-modules over GF(p) on which C_q acts non-trivially are of dimension k where k is the smallest positive integer such that $q \mid p^k - 1$; and the group extensions of these modules by C_q in the obvious way are isomorphic (M.F. Newman [19] Theorem 6.4). Thus in a way similar to what follows some progress may be made in the more general case.

In our case, $q \mid p - 1$, we have $k = 1$ whence N_i is either trivial or a p-cycle for $i = 1, 2$. (Alternatively this can be deduced from the following two facts: (1) the irreducible C_q-modules over $GF(p)$ are absolutely irreducible when $q \mid p - 1$ (Curtis and Reiner [2], Corollary 70.24 , p. 475); (2) the absolutely irreducible C_q-modules over $GF(p)$ are one-dimensional if $p \neq q$ (M. Hall, Jr [5], p. 267).)

Therefore if F is non-abelian it is not difficult to see that it must be the reduced free group of rank 2 of $\underline{\underline{N}}$. For if F_2 denotes the latter group then F_2 can be defined by generators and relations as follows:

$$F_2 = \{a,b \mid a^p = b^p = [a,b]^p = e \ ; [a,b,a] = [a,b,b] = e\} \ .$$

We shall retain the symbols a,b for generators of F_2 . If F is non-trivial abelian then it is cyclic of order p .

The set of all solutions mod p of $x^q \equiv 1 \bmod p$, generates a cyclic group of order q , the unique subgroup of order q of the cyclic multiplicative group of order $p - 1$ of $GF(p)$. Let r be a generator of this q-cycle, $0 < r < p$, $r^q \equiv 1 \mod p$. Suppose g generates a p-cycle and δ a q-cycle C_q. Then for each i, $0 \leq i \leq q - 1$, a C_q-module is formed from $gp\{g\}$ by defining the action of δ on g to map g onto g^{r^i}. In this way the q inequivalent irreducible C_q-modules over $GF(p)$ are obtained.

However, as stated above, if the $q - 1$ irreducible C_q-modules on which C_q acts non-trivially are extended by C_q in the obvious way then the resulting groups are isomorphic. Thus if we consider the groups H_1, H_2 defined by

$$H_1 = gp\{g_1, \delta_1 \mid g_1^p = \delta_1^q = 1 , g_1^{\delta_1} = g_1^r\} \quad ;$$

$$H_2 = gp\{g_2, \delta_2 \mid g_2^p = \delta_2^q = 1 , g_2^{\delta_2} = g_1^{r^i} \} ,$$

where $0 < i \leq q - 1$, then the mapping

$$\delta_2 \longrightarrow \delta_1^i , \qquad g_2 \longrightarrow g_1 ,$$

can clearly be extended to an isomorphism between H_2 and H_1.

The stage is now set for the following theorem.

6.5 <u>Theorem</u>. If the critical group G has a non-abelian Sylow p-subgroup then it is isomorphic to one of the following groups:

$$G_1 = \mathrm{gp}\{F_2, \alpha, \beta \mid [\alpha,\beta] = 1 = \alpha^q = \beta^q;\ a^\alpha = a^r,\ a^\beta = a;$$
$$b^\alpha = b,\ b^\beta = b^r\} \quad ;$$

$$G_2^i = \mathrm{gp}\{F_2, \gamma \mid \gamma^q = 1,\ a^\gamma = a^r,\ b^\gamma = b^{r^i}\} \quad ,$$
$$i = 0,1,\ldots,q-1 \ ;$$

and F_2 . All of these groups are critical and all are 2-generator except G_2^1 : $d(G_2^1) = 3$. (In what follows it will always be clear from the context which group G_2^i the group F_2 is intended to represent a subgroup of.)

6.6 Remark. From the preceding statements and arguments it follows that if G has an abelian Sylow p-subgroup then it is isomorphic to H_1 , C_p or C_q . (H_1 is clearly critical.)

Proof of 6.5. Up to isomorphism the groups G_2^i , $0 \le i \le q-1$, are the only possible non-trivial extensions of F_2 by C_q . This is proved as follows. Suppose F_2' , a p-cycle, is generated by d and, in anticipation of its criticality, denote any extension of F_2 by a q-cycle L , by G . Then $F_2/F_2' = N_1 \times N_2$ where N_1 and N_2 are cyclic of order p and normal in G/F_2' . Let xF_2' and yF_2' be any cosets generating N_1 and N_2 respectively. Then $C_p \times C_p \cong \mathrm{sgp}\{x,d\} \lhd G$. Hence, applying Maschke's Theorem to the L-module $\mathrm{sgp}\{x,d\}$, there exists a direct complement $\mathrm{sgp}\{a\}$ say, of $\mathrm{sgp}\{d\}$, which is also normalized by L . Similarly

$\mathrm{sgp}\{y,d\}$ is split into a direct sum of irreducible L-modules $\mathrm{sgp}\{d\}$ and $\mathrm{sgp}\{b\}$ say. (The subgroup $\mathrm{sgp}\{d\} = F_2'$ is the monolith of G.) Then by choosing a generator γ of L appropriately (as above in arranging an isomorphism between H_2 and H_1) we obtain one of the groups G_2^i as defined in the theorem.

A similar argument yields that G_1 is, up to isomorphism, the only possible extension of F_2 by $C_q \times C_q$.

Thus G_1, G_2^i, $i = 0,\ldots,q-1$, and F_2 are the only candidates with non-abelian Sylow p-subgroups for being critical.. It remains to prove their criticality and to find the minimum number of generators for each.

Obviously F_2 is critical and $d(F_2) = 2$. By Paul M. Weichsel [22] it is the only non-abelian critical group in $\underline{\underline{N}}$. For all i, $0 \leq i \leq q-1$, the proper factors of G_2^i are metabelian. Yet for $i \neq 0$ G_2^i is not metabelian because $[a,\gamma] = a^{r-1}$ and $[b,\gamma] = b^{r^i-1}$ do not commute unless $i = 0$. The group G_2^0 is critical because its proper factors satisfy the law $[x_1^q, x_2^q, x_3^p] = e$ whereas G_2^0 does not, since, in G_2^0,

$$[a^q, b^q, \gamma^p] = [[a,b]^{q^2}, \gamma^p]$$

$$= [a^{r^p}, b]^{q^2}[a,b]^{-q^2} = [a,b]^{q^2(r^p-1)} \neq e .$$

Next $d(G_2^i)$ is determined for $0 \leq i \leq q-1$. If $i \neq 1$, G_2^i is generated by ab and γ: for $(ab)^\gamma = a^r b^{r^i}$

and since

$$[a^r b^{r^i}, ab] = [a,b]^{r-r^i} .$$

it follows that ab and $(ab)^\gamma$ generate F_2 if and only if $r^i \not\equiv r \bmod p$; and $r^i \equiv r \bmod p$ precisely when $i = 1$. The group G_2^1 is the strictly 3-generator critical group promised in the introduction. For if G_2^1 is a 2-generator group then some pair $\{\gamma u, v\}$, $u,v \in F_2$, will do to generate it. There exists $w \in F_2$ such that $w^{r-1} = u$ since $(r-1,p) = 1$. Thus $\gamma u = \gamma w^r w^{-1} = \gamma w^\gamma w^{-1} = w\gamma w^{-1}$ and $G_2^1 = \mathrm{sgp}\{\gamma, v'\}$ where $v' = w^{-1}vw$. Suppose $v' \equiv a^{i_1}b^{i_2}$ modulo F_2' . Then

$$v'^\gamma \equiv a^{i_1 r} b^{i_2 r} \equiv (a^{i_1}b^{i_2})^r \text{ modulo } F_2' .$$

Hence, modulo F_2' , $\mathrm{sgp}\{\gamma u, v\}$ is the extension of C_p by C_q while G_2^1/F_2' is an extension of $C_p \times C_p$ by C_q . Thus a contradiction has been reached and $d(G_2^1) = 3$.

Finally the criticality of G_1 is proved.

First we show that $G_2^i \not\cong G_2^j$ for $i \neq j$, $0 \leq i , j \leq q - 1$. Since G_2^0 is metabelian, whereas for $i \neq 0$ G_2^i is not; and since $d(G_2^1) = 3$, while $d(G_2^i) = 2$ for $i \neq 1$, it may be assumed that $2 \leq i , j \leq q - 1$. Write K_2^i for the factor group by the monolith of G_2^i , $2 \leq i \leq q - 1$. Then K_2^i may be presented as follows:

$$K_2^i = \{a_i, b_i, \gamma_i \mid [a_i, b_i] = e = a_i^p = b_i^p = \gamma_i^q ;$$
$$a_i^{\gamma_i} = a_i^r, b_i^{\gamma_i} = b_i^{r^i}\} ,$$

where a_i , b_i , γ_i are suitable cosets. The only cyclic normal subgroups of K_2^i $(2 \leq i \leq q - 1)$ are $\mathrm{sgp}\{a_i\}$ and $\mathrm{sgp}\{b_i\}$. Therefore any isomorphism $\theta : K_2^i \longrightarrow K_2^j$, must be the extension of one of the following mappings:

$$(1) \left\{ \begin{array}{l} \gamma_i \longrightarrow \gamma_j^{s_1} x_j \\ a_i \longrightarrow a_j^{s_2} \\ b_i \longrightarrow b_j^{s_3} \end{array} \right. \qquad (2) \left\{ \begin{array}{l} \gamma_i \longrightarrow \gamma_j^{s_1} x_j \\ a_i \longrightarrow b_j^{s_2} \\ b_i \longrightarrow a_j^{s_3} \end{array} \right. ,$$

where $x_j \in \mathrm{sgp}\{a_j, b_j\}$ and $0 < s_1, s_2, s_3 < p$. Suppose θ is the extension of the mapping (1). In this case

$$(a_i^{\gamma_i})\theta = a_i^r\theta = (a_i\theta)^r = a_j^{rs_2} ,$$

and

$$(a_i^{\gamma_i})\theta = (a_i\theta)^{\gamma_i\theta} = (a_j^{s_2})^{\gamma_j^{s_1}x_j} = a_j^{r^{s_1}s_2} .$$

Therefore $s_1 = 1$. Also

$$(b_i^{\gamma_i})\theta = (b_i^{r^i})\theta = b_j^{s_3 r^i} ,$$

and

$$(b_i^{\gamma_i})\theta = (b_i\theta)^{\gamma_i\theta} = b_j^{s_3 r^j} .$$

If $i \neq j$ the last two lines give us a contradiction. The mapping (2) can be similarly disposed of. Hence $K_2^i \not\cong K_2^j$ if $i \neq j$, $2 \leq i,j \leq q-1$, and this, together with the remarks on G_2^0 and G_2^1, proves that $G_2^i \not\cong G_2^j$ for $i \neq j$ and $0 \leq i,j \leq q-1$.

By Lemma 6.8 of the following section, and Corollary 6.4, G_2^1 can be embedded in some critical 2-generator group in $\underline{\underline{N}}\,\underline{\underline{A}}_q$. Thus, since obviously $|G_2^i| = |G_2^j|$ for all i,j and by the above $G_2^i \not\cong G_2^j$ for $i \neq j$, the only candidate is G_1 which must therefore be a 2-generator critical group. This completes the proof of the theorem.

The lattice of subvarieties of $\underline{\underline{N}}\,\underline{\underline{A}}_q$

The notation of the preceding section is retained. For convenience of reference a few simple facts which are used in the determination of the subvariety lattice of $\underline{\underline{N}}\,\underline{\underline{A}}_q$, are collected into the following lemma.

6.7 Lemma. The groups G_2^i, $0 \leq i \leq q-1$,

(1) are pairwise non-isomorphic ;

(2) have the same order ;

(3) can be embedded as subgroups of G_1 ;

(4) have the same (isomorphism classes of) proper critical factors.

<u>Proof</u>. Properties (1) and (2) have been dealt with in the proof of Theorem 6.5. Property (4) follows in part from Remark 6.6 and the fact that H_1, C_p and C_q can all be embedded in G_2^i, $0 \leq i \leq q-1$. The only other possible proper critical factor is F_2 which is a factor of all G_2^i. Property (3) is immediate from the fact that the subgroup $\mathrm{sgp}\{F_2, \alpha\beta^i\}$ of G_1 as defined above, is isomorphic to G_2^i, $0 \leq i \leq q-1$.

The final result needed is a corollary of the following lemma.

6.8 <u>Lemma</u>. Let $\underline{\underline{V}}$ be any subvariety of $\underline{\underline{N}}\,\underline{\underline{A}}_q$ containing G_2^i for some fixed i, $0 \leq i \leq q-1$. Let $F_3(\underline{\underline{V}})$ denote the free group of rank 3 of $\underline{\underline{V}}$. Then any normal subgroup H of $F_3(\underline{\underline{V}})$ such that $F_3(\underline{\underline{V}})/H \cong G_2^i$, is complemented in $F_3(\underline{\underline{V}})$, and hence G_2^i can be embedded in $F_3(\underline{\underline{V}})$.

<u>Proof</u>. Since $F_3(\underline{\underline{V}}) \in \underline{\underline{N}}\,\underline{\underline{A}}_q$, it is a splitting extension of its normal Sylow p-subgroup by a Sylow q-subgroup. Let P be the normal Sylow p-subgroup of $F_3(\underline{\underline{V}})$. Let Q ($\cong C_q \times C_q \times C_q$ since $F_3(\underline{\underline{V}})$ is of rank 3) be any Sylow q-subgroup and let $\gamma \in Q \setminus H$.

Now P/P' is elementary abelian and can be considered as a Q-module over $GF(p)$. By Maschke's Theorem the Q-submodule $(H \cap P)P'/P'$ is directly complemented in P/P' by a Q-submodule K say. We shall prove that K is 2-dimensional.

The factor group $PH/H \cong P/H \cap P$ is the Sylow p-subgroup of $F_3(\underline{\underline{V}})/H \cong G_2^i$. Therefore $P/H \cap P \cong F_2$, the free group of rank 2 of $\underline{\underline{N}}$. It follows that

$$\frac{P/H\cap P}{P'(H\cap P)/H\cap P} \cong C_p \times C_p .$$

But this factor group is isomorphic to $P/P'(H\cap P)$, and

$$K \cong \frac{P/P'}{(H\cap P)P'/P'} \cong \frac{P}{P'(H\cap P)},$$

and so K is 2-dimensional.

Directly decompose K into its one-dimensional irreducible Q-submodules K_1 and K_2 and suppose xP' and yP' are cosets generating K_1 and K_2 respectively. Then $\operatorname{sgp}\{P',x\}$ is elementary abelian and is a Q-module of which P' is a Q-submodule. Again by Maschke's Theorem P' is directly complemented in $\operatorname{sgp}\{P',x\}$ by a one-dimensional Q-submodule generated by a say, $a \in P$. In other words an element a can be chosen from the coset xP' such that $\operatorname{sgp}\{a\}$ is normalized by Q. Similarly there is an element b in yP' such that Q normalizes $\operatorname{sgp}\{b\}$. Thus $\operatorname{sgp}\{a,b\}$ is normalized by Q and so in particular by $\operatorname{sgp}\{\gamma\}$.

Let K^* denote the complete inverse image of K under the natural homomorphism $P \longrightarrow P/P'$. Then clearly

$K^* \cap H < P'$. If $[a,b] \in H$ then it is easy to see that $P' < H$, contrary to $P/H \cap P \cong F_2$. Therefore $sgp\{a,b\}$ is a complement of $H \cap P$ in P. It follows easily, by the choice of a and b, that $sgp\{\gamma,a,b\}$ complements H in $F_3(\underline{V})$ and hence that $sgp\{\gamma,a,b\} \cong G_2^i$. Thus the lemma is proved.

Let $\underline{U}$ denote the variety generated by all the proper factors of G_2^i. By 6.7 (4) this is the same for $0 \leq i \leq q-1$.

6.9 Corollary. Let S be the set $\{G_2^i \mid 0 \leq i \leq q-1\}$. Then the lattice of subvarieties of $var(S)$ which contain $\underline{U}$ is isomorphic to the lattice of subsets of S under the mapping defined by

$$var(S_1) \longrightarrow S_1 \ , \quad \phi \neq S_1 \subseteq S \ ;$$

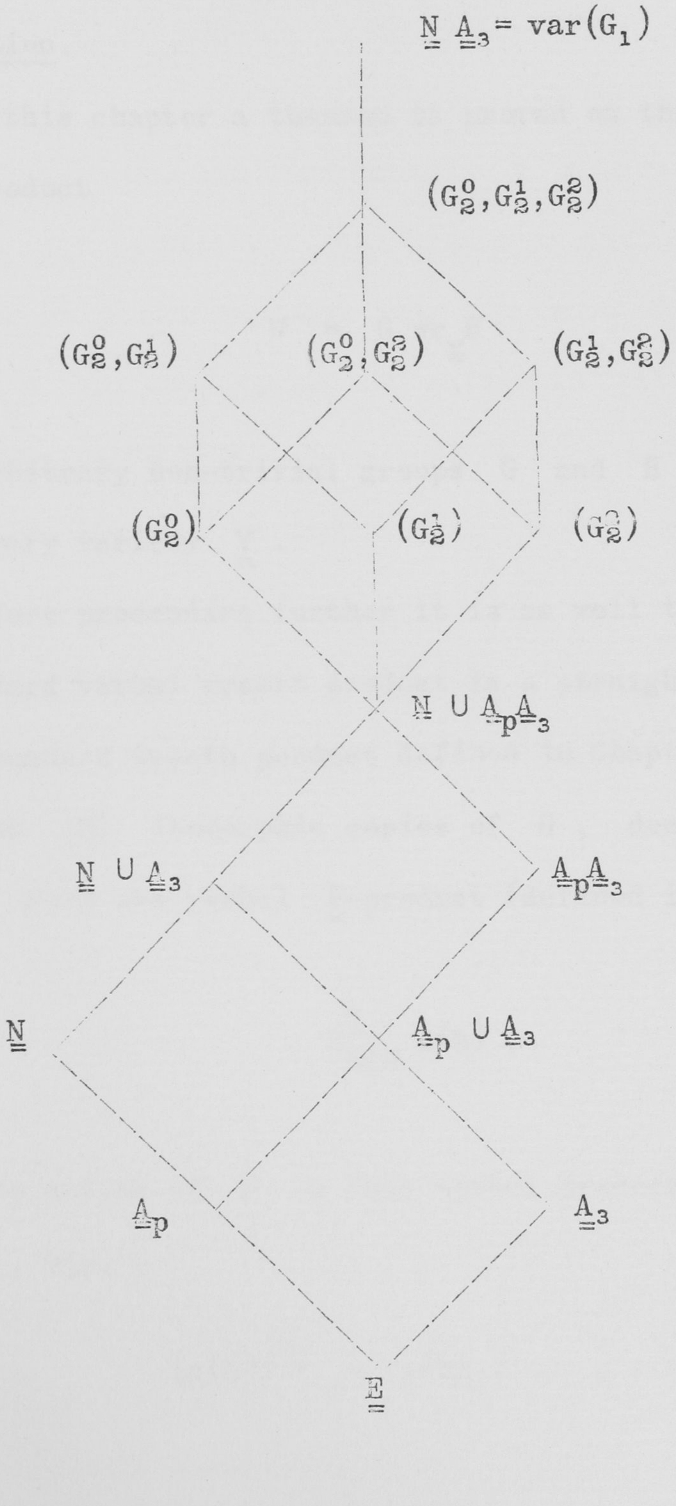
$$\underline{U} \longrightarrow \phi \ .$$

Proof. It suffices to prove that for each i, $0 \leq i \leq q-1$,

$$G_2^i \notin var(S \setminus \{G_2^i\}) \ .$$

Suppose on the contrary that for some fixed i, $G_2^i \in var(S \setminus \{G_2^i\}) = \underline{V}$ say. Then G_2^i is a factor group of $F_3(\underline{V})$ since $d(G_2^i) \leq 3$, and therefore by 6.8 can be embedded in $F_3(\underline{V})$. Now the latter group can be embedded in a finite direct product D of groups in $S \setminus \{G_2^i\}$ and hence so can G_2^i . Suppose $G_2^i \cong A \leq D$. Then G_2^i is in the variety generated by the projections of A into the critical direct factors of D . But by 6.7 (4) and the criticality of G_2^i one of these projections must be isomorphic to G_2^i . This however is excluded by (1) and (2) of 6.7 and the supposition is false. This proves the corollary.

The lattice of subvarieties of $\underline{\underline{U}} = \underline{\underline{N}} \cup \underline{\underline{A}}_p \underline{\underline{A}}_q$ is easily sorted out : by Remark 6.6 the only critical groups in $\underline{\underline{N}} \cup \underline{\underline{A}}_p \underline{\underline{A}}_q$ with abelian Sylow p-subgroups are H_1 , C_p and C_q, and as shown earlier F_2 is the only non-abelian critical group in $\underline{\underline{N}}$. We have therefore effectively obtained the structure of the subvariety lattice of $\underline{\underline{N}}\,\underline{\underline{A}}_q$. It is drawn on the following page (104) for $q = 3$ and any prime p such that $3 \mid p - 1$. The symbols (G_2^2) , (G_2^0, G_2^1) etc. are meant to denote the varieties generated by the group or groups contained in the brackets. The symbol $\underline{\underline{E}}$ denotes the trivial variety.


$\underline{\underline{N}}\ \underline{\underline{A}}_3 = var(G_1)$
(G_2^0, G_2^1, G_2^2)
(G_2^0, G_2^1)
(G_2^0, G_2^2)
(G_2^1, G_2^2)
(G_2^0)
(G_2^1)
(G_2^2)
$\underline{\underline{N}} \cup \underline{\underline{A}}_p\underline{\underline{A}}_3$
$\underline{\underline{N}} \cup \underline{\underline{A}}_3$
$\underline{\underline{A}}_p\underline{\underline{A}}_3$
$\underline{\underline{N}}$
$\underline{\underline{A}}_p \cup \underline{\underline{A}}_3$
$\underline{\underline{A}}_p$
$\underline{\underline{A}}_3$
$\underline{\underline{E}}$

C H A P T E R 7.

A THEOREM ON THE STANDARD VERBAL WREATH PRODUCT.

Introduction.

In this chapter a theorem is proved on the standard verbal wreath product

$$W = G \operatorname{wr}_{\underline{\underline{V}}} H$$

of two arbitrary non-trivial groups G and H with respect to an arbitrary variety $\underline{\underline{V}}$.

Before proceeding further it is as well to give its definition. The standard verbal wreath product is a straightforward generalization of the standard wreath product defined in Chapter 2.

Take $|H|$ isomorphic copies of G, denoted by $G(h)$, $h \in H$. Form the verbal $\underline{\underline{V}}$-product (defined in Chapter 0)

$$\prod_{h\in H}^{\underline{\underline{V}}} G(h) .$$

Define the action of H on this verbal product by its action on $G(h)$, viz. :

7.1 $$(g(h))^{h_1} = g(hh_1) , \quad g \in G , \quad h,h_1 \in H .$$

7.2 <u>Definition.</u> The standard verbal $\underline{\underline{V}}$-wreath product of G by H is the splitting extension of $\prod_{h\in H}^{\underline{\underline{V}}} G(h)$ by H defined by 7.1. That is

$$G \operatorname{wr}_{\underline{\underline{V}}} H = \operatorname{gp}\{\prod_{h\in H}^{\underline{\underline{V}}} G(h), H \mid (g(h))^{h_1} = g(hh_1),\ g \in G,\ h, h_1 \in H\} .$$

<u>Remark.</u> The notation used for this wreath product differs from that used in Part I. The reason is that here we are concerned with standard wreath products only for which the notations $G \operatorname{wr} H$ and $G \operatorname{wr}_{\underline{\underline{V}}} H$ are more common and easier to type. The symbol $G \wr H$ of Philip Hall [6] used in Part I, allowed a uniform notation for the various standard and non-standard wreath products occurring there.

Since our only concern from now on will be with the standard verbal wreath product, the adjective "standard" will, for brevity, be dropped.

Since it requires some preliminary definitions, the statement of the theorem is relegated to the next section. The theorem is a generalization of Theorem 4.1 of Peter M. Neumann [18] and part of the proof is a generalization of the proof in [18].

The theorem is applied in Chapter 8 in the special case $G \in \underline{\underline{V}}$ (see Corollary 7.15) to obtain a lower bound for $\ell(\underline{\underline{N}}\,\underline{\underline{A}}_n)$. For this case a simpler proof of the theorem is possible; however the full theorem is of independent interest. For the case when

$\underline{\underline{V}}$ is an abelian variety (i.e. the case of the standard wreath product of Chapter 2) it has found applications in Part I (as Lemma 2.5) and in the above-mentioned paper [18].

Statement of the theorem.

First some definitions are necessary. Suppose F is a free group on a countably infinite set $\{x_1, x_2, \ldots\}$ of free generators. If $\underline{\underline{V}}$ is any variety, denote by V the corresponding fully invariant subgroup of F and for any group G let $V(G)$ denote the verbal subgroup of G determined by V. Thus $\underline{\underline{V}}$ is the class of all groups G for which $V(G) = E$. Let D be the derived group of F. V and D are verbal subgroups of F and therefore $V \cap D$ is fully invariant and hence also a verbal subgroup, since F is free. It is easy to verify that, for any group G,

$$(V \cap D)(G) \leq V(G) \cap D(G) .$$

In view of subsequent arguments it is worth while pointing out that B.H. Neumann [16] has found an example where the inequality is strict. (Cf. also S. Moran [15].) Finally the fact that $V(G\theta) = V(G)\theta$ for every homomorphism θ, will be used without comment.

Returning to Definition 7.2, henceforth $\prod^{\underline{\underline{V}}}_{h \in H} G(h)$ and H will be regarded as subgroups of W. As in Part I, write $B(W)$ for the "base group" $\prod^{\underline{\underline{V}}}_{h \in H} G(h)$ of W.

We are now in a position to state the theorem.

7.3 Theorem. (Cf. Lemma 2.11.) Let $H_1 > E$ be normal in H, T_1 be any transversal for H_1 in H, and K denote the verbal product $\overset{\underline{\underline{V}}}{\Pi}_{t \in T_1} G(t) < B(W)$. Then

$$H_1^W \cap B(W) = [H_1, B(W)] =$$

$$\{\alpha_1^{h_1}\alpha_2^{h_2}\ldots\alpha_r^{h_r} \mid r \geq 1 ;\ h_i \in H_1,\ \alpha_i \in K,\ \text{for } i = 1,\ldots,r ;$$

$$h_j \neq h_{j+1} \text{ for } j = 1,\ldots,r-1 ;\ \alpha_1\alpha_2\ldots\alpha_r \in (V \cap D)(K)\} .$$

Proof of the theorem.

The proof consists of a string of lemmas. The first two, 7.4 and 7.5, reduce the problem to the case $H_1 = H$. Lemma 7.4 is well-known and simple to prove. The proof is as in the case of the ordinary standard wreath product (see e.g. the proof of Theorem 5.4 of [17]) and is omitted.

7.4 Lemma. If H_2 is any subgroup of H and T_2 is any left transversal for H_2 in H, then

$$H_2.B(W) \cong (\overset{\underline{\underline{V}}}{\Pi}_{t \in T_2} G(t)) \operatorname{wr}_{\underline{\underline{V}}} H_2 .$$

If, in the right hand side, we interpret the action of H_2 on $\overset{\underline{\underline{V}}}{\Pi}_{t \in T_2} G(t)$ as it is in W, we may (and shall in future) replace $\cong$ by $=$.

7.5 Lemma. The normal closure of H_1 in W is $H_1[H_1,B(W)]$.

Proof. The subgroup $[H_1,B(W)]$ of W is normalized by $B(W)$: for, if $h_1 \in H_1$, $b,b_1 \in B(W)$, then

$$[h_1,b]^{b_1} = [h_1,b_1]^{-1}[h_1,bb_1] \in [H_1,B(W)] .$$

Also, $[H_1,B(W)]$ is normalized by H since $H_1 \leq H$ and $B(W) \lhd W$. Hence $[H_1,B(W)]$ is normal in $H.B(W) = W$.

The subgroup $H_1[H_1,B(W)]$ is obviously normalized by H . Also, if $b \in B(W)$, $h_1 \in H_1$, then

$$h_1^b = h_1[h_1,b] \in H_1[H_1,B(W)] .$$

Hence $H_1[H_1,B(W)]$ is normal in W and must be the normal closure of H_1 in W .

7.6 Corollary. The normal closure of H_1 in W is the same as its normal closure in $H_1B(W)$.

This follows at once from Lemma 7.5. This corollary together with Lemma 7.4 allows us to assume that $H_1 = H$. The following few lemmas are concerned with this case and with a particular verbal wreath product.

Consider the particular verbal wreath product $W^* = G \text{ wr}_* H$ (which is in fact isomorphic to the free product $G * H$ - but this will not be needed) since the more general wreath product W is a factor group of it. The base group of W^* is the free product $\Pi^*_{h\in H} G(h), = \Pi^* G(h)$ say. Now by the definition (0.2) of the verbal product,

$$B(W) = \overset{\underline{\underline{V}}}{\Pi}_{h\in H} G(h) = \Pi^* G(h)/V(\Pi^* G(h)) \cap C \quad ,$$

where C is the cartesian subgroup of the free product $\Pi^* G(h)$. For brevity, write $\Lambda = V(\Pi^* G(h)) \cap C$, so that

$$B(W) = \Pi^* G(h)/\Lambda ; \qquad W = W^*/\Lambda \quad .$$

Any element of $\Pi^* G(h)$ can be written uniquely in the normal form $g_1(h_1)\ldots g_r(h_r)$ where $g_i \in G$, $h_i \in H$, $i = 1,\ldots,r$; $h_j \neq h_{j+1}$, $j = 1,\ldots,r-1$. Write

$$X = \{g(e)g^{-1}(h) \mid g \in G,\ h \in H\}$$

$$= \{[g^{-1}(e),h] \mid g \in G,\ h \in H\} \subset \Pi^* G(h) \quad .$$

Then the following is true.

7.7 <u>Lemma</u>. If $g_1(h_1)\ldots g_r(h_r)$, $g_i \in G$, $h_i \in H$, $h_j \neq h_{j+1}$

is any element of $\Pi^* G(h)$, then the element

$$b = (g_1 \ldots g_r)^{-1}(h_1)\, g_1(h_1) \ldots g_r(h_r)$$

lies in $\mathrm{sgp}(X)$.

Proof. Any element of the form $g^{-1}(h')g(h'')$, $g \in G$, $h', h'' \in H$, belongs to $\mathrm{sgp}(X)$ since

$$g^{-1}(h')g(h'') = (g^{-1}(1)g(h'))^{-1} g^{-1}(1)g(h'') \in \mathrm{sgp}(X) \quad .$$

Write

$$x_1 = g_r^{-1}(h_r)g_r(h_{r-1}) \quad ,$$

$$x_2 = (g_{r-1}g_r)^{-1}(h_{r-1})(g_{r-1}g_r)(h_{r-2}) \quad ,$$

$$x_{r-1} = (g_2 g_3 \ldots g_r)^{-1}(h_2)(g_2 g_3 \ldots g_r)(h_1) \quad .$$

Then

$$x_1 x_2 \ldots x_{r-1} = b^{-1} \quad ,$$

and since $x_1 x_2 \ldots x_{r-1} \in \mathrm{sgp}(X)$ by the preceding remark, the required result is obtained.

7.8 Corollary. In W^* , $[H, \Pi^* G(h)] = \mathrm{sgp}(X)$.

Proof. It is obvious from the definition of X that $[H, \Pi^* G(h)] \geq \mathrm{sgp}(X)$. To prove the reverse inclusion, it will be shown that every generator $[h',b]$ of $[H,\Pi^* G(h)]$, $h' \in H$, $b \in \Pi^* G(h)$, lies in $sgp(X)$. Suppose

$$b = g_1(h_1)\ldots g_r(h_r)$$

is in the normal form. Then

$$\begin{aligned}[h',b] &= (g_1(h_1h')\ldots g_r(h_rh'))^{-1} g_1(h_1)\ldots g_r(h_r) \\ &= g_r^{-1}(h_rh')\ldots g_1^{-1}(h_1h')g_1(h_1)\ldots g_r(h_r) .\end{aligned}$$

Thus, by Lemma 7.7 , if both sides are multiplied on the left by $(g_r^{-1}\ldots\ldots g_1^{-1} g_1\ldots\ldots g_r)(h_rh') = e$, we obtain

$$[h',b] \in sgp(X) ,$$

and the proof is complete.

The next lemma, which is more difficult, is concerned with the free product $\Pi^* G(h)$ only.

7.9 <u>Lemma</u>. The epimorphism $\theta : \Pi^* G(h) \longrightarrow G$ defined by $g(h)\theta = g$, $g \in G$, $h \in H$, (that is, θ amalgamates the $G(h)$) maps $\Lambda = V(\Pi^* G(h)) \cap C$ onto $(V \cap D)(G)$.

This is slightly surprising as one might expect on the face of it that $\Lambda\theta = V(G) \cap D(G)$ which, as remarked above, is not always the same as $(V \cap D)(G)$.

For proving this lemma a preliminary result is needed which for clarity is stated and proved separately. First a definition is given (Higman [8]).

7.10 Definition. A commutator in $x_1^{\pm 1}, x_2^{\pm 1}, \ldots,$ or with entries from $\{x_1^{\pm 1}, x_2^{\pm 1}, \ldots\}$ is defined as follows. The elements x_i, x_i^{-1} are commutators in the elements of $\{x_1^{\pm 1}, x_2^{\pm 1}, \ldots\}$ of weight 1 and if y and z are commutators in $x_1^{\pm 1}, x_2^{\pm 1}, \ldots,$ of weight r and s respectively then $[y,z]$ is a commutator in the variables, of weight $r + s$. Only commutators formed in this way are allowed.

To say that a word (element) in F "involves" x_i will mean that the word contains x_i or x_i^{-1} when written in reduced form. A commutator with entries from $\{x_1^{\pm 1}, x_2^{\pm 1}, \ldots\}$ that involves x_i , takes the value e when x_i is replaced by e . Also if the commutator y involves x_i , then so does $[y,z]$.

7.11 Lemma.† The verbal subgroup $V \cap D$ is generated by the set of all those words $w = w(x_1, \ldots, x_n)$ in $V \cap D$, each of which can be written as a product $c_1 \ldots c_s$ say, of commutators

---

† This lemma was suggested by Professor Hanna Neumann as a correction of my original proof of Lemma 7.9.

of weight >1 with entries from $\{x_1^{\pm 1}, x_2^{\pm 1}, \ldots\}$, such that that there exist two distinct subscripts j and k with the property that each commutator c_i, $1 \leq i \leq s$, involves both x_j and x_k($(1 \leq j, k \leq \ell)$.

<u>Proof</u>. The proof is by induction on the number of variables a word in $V \cap D$ involves. If this number is 2, the word already has the required form. Suppose that words in $V \cap D$ involving fewer than ℓ variables are products of words of the form w, and let $v = v(x_1, \ldots x_\ell)$ be any element of $V \cap D$ which involves all of $x_1, \ldots x_\ell$. Since $v \in D$, it can be written as a product of commutators of weight >1 in $x_1^{\pm 1}, \ldots, x_\ell^{\pm 1}$: $v = c_1' \ldots c_t'$ say. If such a commutator y_1 involves x_i, and y_2 does not, then using the identity $y_1 y_2 = y_2 y_1 [y_1, y_2]$ we can see to it that the commutator not involving x_i comes before commutators involving x_i. Hence we can write

$$v = v_1 v_2 v_3 v_4$$

where the v_i, $i = 1,2,3,4$, are products of commutators of weight >1 in the x_i's and their inverses, and in v_1 all factors involve neither x_1 nor x_2 ; in v_2 all involve x_1 but not x_2 ; in v_3 all involve x_2 but not x_1 ; and the factors of v_4 all involve both x_1 and x_2. If we put successively $x_1 = x_2 = e$; $x_1 = e$; $x_2 = e$, in v, we see that $v_1, v_2, v_3, v_4 \in V \cap D$. Now v_1 , v_2 and v_3 involve at most $\ell - 1$ variables and so, by the inductive hypothesis, they are products of words in $V \cap D$ of the right form. The element v_4 is already of the required form. Obviously no generality has been lost by working with the particular ℓ variables $x_1, \ldots, x_\ell$, and the proof is complete.

<u>Proof of 7.9.</u> Let $w = w(x_1, \ldots, x_\ell)$ be as in the statement of Lemma 7.11 : $w = c_1 \ldots c_s$ and the c_i simultaneously involve the variables x_j and x_k . Since $H \neq E$, there exist $h', h'' \in H$ for which $h' \neq h''$. Substitute in w any ℓ elements $g_1, \ldots g_\ell \in G$, for $x_1, \ldots, x_\ell$ respectively. The resulting element is in $(V \cap D)(G)$. On the other hand, if we substitute $g_i(h')$ for x_i , $i \neq k$, and $g_k(h'')$ for x_k , the form of w ensures that the resulting element is in Λ . Now

$$w(g_1(h'), \ldots, g_k(h''), \ldots, g_\ell(h'))\theta \; = \; w(g_1, \ldots, g_\ell) \quad ,$$

and since, by 7.11, all such elements $w(g_1, \ldots, g_\ell)$ generate $(V \cap D)(G)$, it follows that

$$\Lambda\theta \; \geq \; (V \cap D)(G) \; .$$

We shall now prove the reverse inclusion.

Let R be the kernel of an epimorphism Ψ from a free group F_1 of suitable rank, onto G : $G \cong F_1/R$. Take $|H|$ isomorphic copies of F_1 denoted by $F_1(h)$, $h \in H$. Then corresponding to Ψ we have, for each $h \in H$, the obvious epimorphism

$$\Psi(h) : F_1(h) \longrightarrow G(h) \quad .$$

Consider the free product $\Pi^*_{h\in H} F_1(h)$, $= \Pi^* F_1(h)$ say. Let φ be the epimorphism from $\Pi^* F_1(h)$ onto $\Pi^* G(h)$, whose restriction to $F_1(h)$ is $\Psi(h)$. It then suffices to prove that

7.12 $$(V \cap D)(\Pi^* F_1(h))\varphi \geq \Lambda \quad .$$

For,

$$(V \cap D)(\Pi^* F_1(h))\varphi\theta = (V \cap D)(G) \quad ,$$

and so, from 7.12,

$$(V \cap D)(G) \geq \Lambda\theta \quad .$$

7.12 is proved as follows. Let $v(\underset{\sim}{x}) = v(x_1,\ldots,x_\ell) \in V$, be such that for some $b_1,\ldots,b_\ell \in \Pi^* G(h)$, we have

$$v(b_1,\ldots,b_\ell) \in C \quad .$$

Every element g of $\Lambda = V(\Pi^* G(h)) \cap C$ is obtainable in this way from some $v(\underset{\sim}{x}) \in V$. Suppose $f_i\varphi = b_i$, where $f_i \in \Pi^* F_1(h)$, $i = 1,\ldots,\ell$. Consider now the element $v(\underset{\sim}{f}) = v(f_1,\ldots,f_\ell)$. Modulo its cartesian, $\Pi^* F_1(h)$ is the direct product of the $F_1(h)$. From this, together with the fact that the complete inverse image of C under φ is the product of the cartesian of $\Pi^* F_1(h)$ and the normal closure in $\Pi^* F_1(h)$ of all $R(h)$, we deduce that we may write

$$v(\underset{\sim}{f}) = d.k_1(h_1)\dots k_t(h_t) \quad ,$$

where d belongs to the cartesian of $\Pi^* F_1(h)$; $k_i \in R$, $h_i \in H$, $i = 1,\dots,t$ and $h_j \neq h_k$ for $j \neq k$.

Next suppose $\{y_\gamma \mid \gamma \in \Gamma\}$ (Γ some index set) is a set of free generators of F_1. Then $\{y_\gamma(h) \mid \gamma \in \Gamma, h \in H\}$ generates $\Pi^* F_1(h)$ freely and the f_i may be regarded as reduced words in these free generators. Consider the subset

$$S = \{y_\gamma(h) \mid \text{for some } i = 1,\dots\ell,\ f_i \text{ involves } y_\gamma(h)\} \quad ,$$

and let $\mu : \mathrm{sgp}(S) \longrightarrow F$ be the monomorphism extending any $(1,1)$ mapping from S into the free generators $\{x_1, x_2, \dots\}$ of F. Then $v(\underset{\sim}{f})\mu \in V$. If, for some fixed j, $1 \leq j \leq t$, we set $x_i = e$ in the word $v(\underset{\sim}{f})\mu$ whenever $x_i\mu^{-1} \notin F_1(h_j)$, then we find that also

$$k_j(h_j)\mu \in V \quad , \quad j = 1,\dots,t \quad .$$

Hence $d\mu \in V$; obviously $d\mu \in D$. Write $d\mu = v'$.

Thus, for each element $g \in \Lambda$ a word $v' \in V \cap D$ has been found such that the substitution $\mu^{-1}\varphi$ gives g : $v'\mu^{-1}\varphi = g$. Now $v'\mu^{-1} = d \in (V \cap D)(\Pi^* F_1(h))$ and hence $(V \cap D)(\Pi^* F_1(h))\varphi \geq \Lambda$ as required. This completes the proof of 7.12 and thence of Lemma 7.9.

7.13 <u>Corollary</u>. For all $h \in H$, $(V \cap D)(G(h))$ is contained in $sgp(X).\Lambda$.

<u>Proof</u>. Corollary 7.8 implies that $sgp(X)$ is normal in W^* . Let a be any element in $(V \cap D)(G)$. Then by Lemma 7.9 there exists an element $g = g_1(h_1)\ldots g_r(h_r)$ in Λ such that $g_1\ldots g_r = a$. By Lemma 7.7 $a^{-1}(h_1)g \in sgp(X)$, whence $a(h_1) \in sgp(X)\Lambda$. The normality of $sgp(X)$ in W^* then gives the stated result.

Finally we return to W for which we obtain the following corollary.

7.14 <u>Corollary</u>. $[H,B(W)] = sgp(X)\Lambda/\Lambda = M$, where M is the set of cosets

$$\{g_1(h_1)\ldots g_r(h_r)\Lambda \mid r \geq 1;\ h_i \in H,\ g_i \in G,\ i = 1,\ldots,r;$$
$$h_j \neq h_{j+1},\ j = 1,\ldots,r-1;\ g_1\ldots g_r \in (V \cap D)(G)\} .$$

<u>Proof</u>. It is easy to see from 7.9 that the set M forms

a subgroup of W. Therefore, since $X\Lambda/\Lambda$, the set of cosets $x\Lambda$, $x \in X$, is contained in M, it follows that $\mathrm{sgp}(X)\Lambda/\Lambda \leq M$. To prove the reverse inclusion write $g = g_1(h_1)\ldots g_r(h_r)$ where $g_1 \ldots g_r \in (V \cap D)(G)$. Then, by Lemma 7.7, $(g_1\ldots g_r)^{-1}(h_1)\, g_1(h_1)\ldots g_r(h_r) \in \mathrm{sgp}(X)$. Therefore $g \in (V \cap D)(G(h_1)).\mathrm{sgp}(X)$ which by 7.13 is contained in $\mathrm{sgp}(X).\Lambda$. The equality

$$[H, B(W)] = \mathrm{sgp}(X)\Lambda/\Lambda$$

is immediate from Corollary 7.8. The proof is now complete.

<u>Proof of Theorem 7.3</u>. The theorem now follows immediately from Lemmas 7.4 and 7.5 and Corollary 7.14.

For the application of Theorem 7.3 in Chapter 8, it is required only in the case $V(G) = E$. In this case, for convenience it is restated as a corollary.

7.15 <u>Corollary</u>. With the same notation as in Theorem 4.1, suppose in addition to the hypotheses of 4.1 that $V(G) = E$. Then the normal closure of the normal subgroup $H_1 > E$ of H (with a transversal T_1 in H) in $W = G \operatorname{wr}_{\underline{V}} H$, is H_1M_1 where

$$M_1 = \{\alpha_1^{h_1}\ldots\alpha_r^{h_r} \mid r \geq 1;\ h_i \in H_1,\ \alpha_i \in \prod_{t\in T_1}^{\underline{V}} G(t),$$
$$i = 1,\ldots r;\ h_j \neq h_{j+1},\ j = 1,\ldots,r-1;\ \alpha_1\alpha_2\ldots\alpha_r = e\}.$$

CHAPTER 8.

A LOWER BOUND FOR $\ell(\underline{\underline{N}}\,\underline{\underline{A}}_n)$.

Introduction.

In Chapter 5 it was stated as a theorem (Theorem 5.1) that if c is the class of a nilpotent variety $\underline{\underline{N}}$ of exponent m and $(m,n) = 1$, then for $c > 1$, $\ell(\underline{\underline{N}}\,\underline{\underline{A}}_n) = c$ while for $c = 1$, $\ell(\underline{\underline{N}}\,\underline{\underline{A}}_n) = 2$. We proved there that for $c > 1$, $\ell(\underline{\underline{N}}\,\underline{\underline{A}}_n) \leq c$ and for $c = 1$, $\ell(\underline{\underline{N}}\,\underline{\underline{A}}_n) = 2$. This chapter is devoted to the proof of the remaining half of Theorem 5.1; viz.

8.1 $$\ell(\underline{\underline{N}}\,\underline{\underline{A}}_n) \geq c \quad \text{for all } c \geq 1 .$$

For any variety $\underline{\underline{V}}$, let $F_k(\underline{\underline{V}})$ denote the reduced free group of rank k of $\underline{\underline{V}}$. The method of proving 8.1 involves certain properties of the verbal wreath product,

$$W_c = F_c(\underline{\underline{N}}) \operatorname{wr}_{\underline{\underline{N}}} F_c(\underline{\underline{A}}_n) .$$

The reduced free group $F_c(\underline{\underline{A}}_n)$ is isomorphic to the direct power C_n^c of the n-cycle C_n. Clearly $W_c \in \underline{\underline{N}}\,\underline{\underline{A}}_n$. We shall show that if $W_c \in \operatorname{var}(F_{c-1}(\underline{\underline{N}}\,\underline{\underline{A}}_n))$ then W_c is not only a factor of, but can be embedded in, a (finite) direct power of $F_{c-1}(\underline{\underline{N}}\,\underline{\underline{A}}_n)$.

Hence by Lemma 0.1 there is a set of normal subgroups of W_c, with trivial intersection, giving rise to factor groups embeddable in $F_{c-1}(\underline{\underline{N}}\,\underline{\underline{A}}_n)$. We prove that for all possible such sets, the factor group of at least one normal subgroup is not so embeddable.

Preliminary results.

In addition to Corollary 7.15 a few lemmas are needed. The following lemma has also been proved by A.L. Šmel'kin in his paper [21]. His proof relies on the main theorem of that paper. The proof following is short and more direct.

8.2 Lemma. Let $\underline{\underline{U}}$ and $\underline{\underline{V}}$ be locally finite varieties of coprime exponents m and n respectively. Then the verbal wreath product

$$W(k) = F_k(\underline{\underline{U}}) \operatorname{wr}_{\underline{\underline{U}}} F_k(\underline{\underline{V}})$$

can be embedded in $F_{2k}(\underline{\underline{U}}\,\underline{\underline{V}})$, the free group of rank $2k$ of $\underline{\underline{U}}\,\underline{\underline{V}}$, for all $k \geq 1$.

Proof. Let F_{2k} be absolutely free on free generators $x_1, \ldots, x_{2k}$. Then, by the Schur-Zassenhaus Theorem and the hypotheses of the lemma, $F_{2k}(\underline{\underline{U}}\,\underline{\underline{V}}) = F_{2k}/U(V(F_{2k}))$ is a splitting extension of $V(F_{2k})/U(V(F_{2k}))$ by $F_{2k}(\underline{\underline{V}}) = F_{2k}/V(F_{2k})$. If we write $F_k = \operatorname{sgp}\{x_1, \ldots, x_k\}$, the same remark applied to $F_k < F_{2k}$ shows that there exists a set $T \subset F_k$ which is a transversal for $V(F_k)$ in F_k and is also, modulo $U(V(F_{2k}))$, a complement of $V(F_k)$ in F_k. Let T_1 be a right Schreier transversal for $V(F_k)$ in F_k (Kurosh [13], Vol. 2, p. 33). Since $V(F_k) = F_k \cap V(F_{2k})$, T and T_1 are subsets of some transversals for $V(F_{2k})$ in F_{2k}. For each $t_1 \in T_1$ there exists a unique $t \in T$ such that

8.3 $$t_1 = a_t t ,$$

where a_t depends on t and $a_t \in V(F_k)$. The mapping $T_1 \longrightarrow T$ so defined is $(1,1)$ and onto.

We now show that one can choose a right Schreier transversal T_2 for $V(F_{2k})$ in F_{2k} which contains the set

$$S = \{t_1 x_i^{\alpha_i} \mid t_1 \in T_1,\ 0 \leq \alpha_i \leq n-1,\ i = k+1,\dots,2k\} ,$$

by the following simple modification of the usual argument. Distinct elements of S lie in different cosets of $V(F_{2k})$ in F_{2k}. For if $(t_1 x_i^{\alpha_i})^{-1} t_1' x_j^{\alpha_j} \in V(F_{2k})$ then by replacing x_i and x_j by e we obtain $t_1^{-1} t_1' \in V(F_k)$, whence $t_1 = t_1'$. Then $x_i^{-\alpha_i} x_j^{\alpha_j} \in V(F_{2k})$. If $i \neq j$ and $\alpha_i \neq 0$ set $x_j = e$; then $x_i^{-\alpha_i} \in V(F_{2k})$. But $0 < \alpha_i \leq n-1$ and $\underline{\underline{V}}$ is of exponent n, so either $i = j$ or $\alpha_i = 0$. In this way one will quickly be led to the preceding assertion. The set S

obviously has the Schreier property. Choose the elements of S as representatives of their cosets. Let Q be the set of cosets not so represented. To find suitable representatives for the cosets in Q we use induction on the smallest length of elements of each coset in Q (as in [13], vol. 2, p. 33). The details are as follows. The identity e belongs to S. Let $w = x_{i_1}^{e_1} \dots x_{i_\ell}^{e_\ell}$, $e_i = \pm 1$, $i = 1, \dots, \ell$, be a reduced word in some coset in Q, of smallest length ℓ in that coset. Suppose the required choice of representatives has already been made for cosets in Q with smaller minimal length. Then the representative v say, of the coset containing $x_{i_1}^{e_1} \dots x_{i_{\ell-1}}^{e_{\ell-1}}$, which may or may not be in Q, has in any case already been chosen. Choose $vx_{i_\ell}^{e_\ell}$ as the representative of the coset containing w. Thus the required T_2 is obtained. (Cf. also M.J. Dunwoody [3].)

A set of free generators of $V(F_{2k})$ is then

$$S_{2k} = \{t_2 x_i (\varphi(t_2 x_i))^{-1} \mid i = 1, \dots, 2k, \ t_2 \in T_2\} \setminus \{e\} ,$$

where $\varphi(g), g \in F_{2k}$, is the element of T_2 representing the coset $gV(F_{2k})$. The set

$$S_k = \{t_1 x_i (\varphi(t_1 x_i))^{-1} \mid i = 1, \dots, k, \ t_1 \in T_1\} \setminus \{e\}$$

is a set of free generators of $V(F_k)$. By the way T_2 was chosen, $S_k \subset S_{2k}$. Write

$$X = \{t_2x_i(\varphi(t_2x_i))^{-1} \mid t_2 = t_1x_i^{n-1}\ ,\ t_1 \in T_1,\ i = k+1,\ldots,2k\}$$

$$= \{(x_i^n)^{t_1^{-1}} \mid t_1 \in T_1,\ i = k+1,\ldots,2k\} \subset S_{2k} \setminus S_k\ .$$

By 8.3,

$$(x_i^n)^{t_1^{-1}} = ((x_i^n)^{t^{-1}})^{a_t^{-1}}\ ,\ a_t^{-1} \in \mathrm{sgp}(S_k)\ .$$

Write

$$Y = \{(x_i^n)^{t^{-1}} \mid t \in T,\ i = k+1,\ldots,2k\}\ .$$

Since $a_t \in \mathrm{sgp}(S_k)$ for all $t \in T$ and Y is obtained from $X \subset S_{2k} \setminus S_k$ by suitable conjugation by the a_t , it follows that $(S_{2k} \setminus X) \cup Y$ is an alternative set of free generators of $V(F_{2k})$.

Modulo $U(V(F_{2k}))$, $\mathrm{sgp}\{x_i^n \mid i = k+1,\ldots,2k\}$ is isomorphic to $F_k(\underline{\underline{U}})$. Because of the way T was chosen, we have, modulo $U(V(F_{2k}))$,

$$\mathrm{sgp}\{T, x_i^n \mid i = k+1,\ldots,2k\}$$

$$\cong \mathrm{sgp}\{x_i^n \mid i = k+1,\ldots,2k\}\ \mathrm{wr}_{\underline{\underline{U}}}\ \mathrm{sgp}\ (T)$$

which is isomorphic $(\mathrm{mod}\ U(V(F_{2k})))$ to $W(k)$. This completes the proof.

8.4 Corollary. If $F_\nu(\underline{\underline{U}}\,\underline{\underline{V}})$ generates $\underline{\underline{U}}\,\underline{\underline{V}}$ for some cardinal ν, then W_k can be embedded in some finite direct power of it.

Proof. Under the hypothesis of the corollary, any finitely generated free group of $\underline{\underline{U}}\,\underline{\underline{V}}$ can be embedded in a suitably large finite direct power of $F_\nu(\underline{\underline{U}}\,\underline{\underline{V}})$. In particular this applies to $F_{2k}(\underline{\underline{U}}\,\underline{\underline{V}})$ and thus to W_k by Lemma 8.2.

The following lemma together with Corollary 8.4 completes the preparation for the proof of 8.1.

8.5 Lemma. If $n = p$, a prime, then every set of normal subgroups of $W_c = F_c(\underline{\underline{N}})\ \mathrm{wr}_{\underline{\underline{N}}}\ F_c(\underline{\underline{A}}_n)$ such that none of the normal subgroups is contained wholly in the base group, has non-trivial intersection.

For the proof a corollary of the following lemma is needed.

8.6 Lemma. (Cf. Higman [8].) Let F_k be absolutely free on $x_1,\ldots,x_k$, $k \geq 2$, and let $\underline{\underline{V}}$ be a nilpotent variety of class c. Denote the subsets of $\{x_1,\ldots,x_k\}$ containing not less than 2 and not more than c elements by $S_1,S_2,\ldots,S_\ell$, and form

$$C(S_i) = \mathrm{sgp}\{\gamma \mid \gamma \text{ a commutator of weight } c \text{ with set of entries precisely } S_i\}\ .$$

Then, if θ is the natural homomorphism $F_k \to F_k/V(F_k)$, we have

$$\mathrm{sgp}\{C(S_i) \mid i = 1,\ldots,\ell\}\theta = C(S_1)\theta \times \ldots \times C(S_\ell)\theta .$$

<u>Proof</u>. It suffices to prove that if

$$\gamma_1\gamma_2\ldots\gamma_\ell \in V(F_k) ,$$

where $\gamma_i \in C(S_i)$, $i = 1,\ldots,\ell$, then

$$\gamma_i \in V(F_k) \text{ for each } i .$$

The proof is by induction on the cardinal of S_i. Let φ_i be the endomorphism of F_k which fixes the elements of S_i and maps all other free generators in $\{x_1,\ldots,x_k\}$ onto e. Then if $|S_i| = 2$,

$$(\gamma_1\gamma_2\ldots\gamma_\ell)\varphi_i = \gamma_1\varphi_i\gamma_2\varphi_i\ldots\gamma_\ell\varphi_i = \gamma_i .$$

Therefore $\gamma_i \in V(F_k)$. Assume as inductive hypothesis that $\gamma_j \in V(F_k)$ for all S_j with $|S_j| \leq s < k$. We may then omit from $\gamma_1\ldots\gamma_\ell$ those γ_j's whose corresponding S_j's contain not more than s elements, and then have the remaining product still belonging to $V(F_k)$. If $|S_i| = s + 1$, an application of φ_i to this smaller product completes the proof of the inductive step and thence the proof of the lemma.

We apply this lemma to the base group $B(W_c)$ of W_c. Suppose $x_1,\dots,x_c$ freely generate $F_c(\underline{\underline{N}})$, the "bottom" group of W_c. Then there exists a commutator of weight c with its set of entries precisely $\{x_1,\dots,x_c\}$ (in the sense of 7.10) which does not reduce to e. Otherwise, since $F_c(\underline{\underline{N}})$ is reduced free, all commutators of weight c would be e and $F_c(\underline{\underline{N}})$ would be nilpotent of class $< c$. Denote such a commutator by $\gamma(x_1,\dots,x_c)$. An obvious set of free generators of $B(W_c)$, which is the free group of rank $|F_c(\underline{\underline{A}}_n)|.c = n^c c$ of $\underline{\underline{N}}$, is $\{x_i(h) \mid i = 1,\dots,c,\ h \in F_c(\underline{\underline{A}}_n)\}$. Then we have the following corollary of Lemma 8.6.

8.7 <u>Corollary</u>. The elements of the set

$$\{\gamma(x_1(h_1),\dots,x_c(h_c)) \mid h_i \in F_c(\underline{\underline{A}}_n)\} ,$$

are all non-trivial of order m_1 where m_1 divides m, the exponent of $\underline{\underline{N}}$, and independent. If $m_1 = qm_2$ where q is a prime, then

$$S = \{(\gamma(x_1(h_1),\dots,x_c(h_c)))^{m_2} \mid h_i \in F_c(\underline{\underline{A}}_n)\}$$

is a basis for $C_q^{n^{c^2}}$ considered as a vector space over $GF(q)$ of dimension n^{c^2}.

<u>Proof of Lemma 8.5</u>. We shall use Philip Hall's well-known theorem (see e.g. M. Hall, Jr. [5], p. 141) the relevant parts of which are stated below for convenience.

8.8 Let G be a soluble group of order rs where $(r,s) = 1$. Then

8.8.1 G has at least one subgroup of order r;

8.8.2 any two subgroups of order r are conjugate;

8.8.3 any subgroup whose order r' divides r, is contained in a subgroup of order r.

Since $\underline{\underline{N}}\,\underline{\underline{A}}_n$ is a soluble, locally finite variety, we may apply this theorem to any of its finitely generated groups.

We are dealing with the case $W_c = F_c(\underline{\underline{N}}) \text{ wr}_{\underline{\underline{N}}} F_c(\underline{\underline{A}}_p)$ where p is a prime. It follows that $|W_c| = p^c t$ where $(p,t) = 1$ and $|B(W_c)| = t$. Since $B(W_c) \triangleleft W_c$, 8.8.2 implies that $B(W_c)$ is the only subgroup of order t, and by 8.8.3 $B(W_c)$ contains every subgroup of order prime to p. Let K be any normal subgroup of W_c not wholly contained in $B(W_c)$. Suppose $|K| = p^{\alpha} t_1$ where $(p,t_1) = 1$. Then by the above, $\alpha > 0$. By 8.8.1, K contains a subgroup of order t_1. This must be $K \cap B(W_c)$. Again by 8.8.1, $K \cap B(W_c)$ is complemented in K by P say, and by 8.8.2, 8.8.3 we may choose P to be a subgroup of the top group $F_c(\underline{\underline{A}}_p) \cong C_p^c$ of W_c.

Thus the truth of the lemma will follow if we show that the intersection of the normal closures in W_c of all p-cycles contained in $F_c(\underline{\underline{A}}_p)$, is non-trivial. This we proceed to do.

Denote the elements of S (defined in Corollary 8.7) by $\gamma_1,\ldots,\gamma_s$ where $s = p^{c^2}$ in the present context. We shall prove that an element of $B(W_c)$ of the form

$$b = \gamma_1^{e_1}\ldots\gamma_s^{e_s}$$

lies in the above-mentioned intersection for some $e_1,\ldots,e_s$ not all congruent to 0 mod q.

Let $\Gamma \cong C_p$ be any p-cycle in the top group $F_c(\underline{\underline{A}}_p)$ of W_c and let T be any transversal for Γ in $F_c(\underline{\underline{A}}_p)$. Then $|T| = p^{c-1}$. Denote by φ_Γ the mapping of $F_c(\underline{\underline{A}}_p)$ into itself which sends each element onto its representative in T. Then let $\varphi'_\Gamma : sgp(S) \longrightarrow sgp(S)$ be the homomorphism obtained by extending the mapping defined by

$$\gamma(x_1(h_1),\ldots,x_c(h_c))\varphi'_\Gamma = \gamma(x_1(h_1\varphi_\Gamma),\ldots,x_c(h_c\varphi_\Gamma)) .$$

By Corollary 7.15 $\gamma_1^{e_1}\ldots\gamma_s^{e_s} \in \Gamma^{W_c}$ if and only if

8.9 $$(\gamma_1^{e_1}\ldots\gamma_s^{e_s})\varphi'_\Gamma = e .$$

If we collect the γ_i's which have become identified under φ'_Γ, then in the collected expression we must have powers of distinct γ_i's congruent to 0 mod q for 8.9 to be satisfied. In this way we are led to $|T|^c = p^{c(c-1)}$ linear homogeneous equations in $e_1,\dots,e_s$ over $GF(q)$. Thus $\gamma_1^{e_1}\dots\gamma_s^{e_s} \in \Gamma^{W_c}$ if and only if $e_1,\dots,e_s$ is a solution of this system of equations. It follows that if b is to lie in the intersection of the normal closures of all p-cycles contained in $F_c(\underline{\underline{A}}_p)$, $e_1,\dots,e_s$ must be a solution simultaneously of the corresponding systems of linear equations. Since there are $p^c-1/p-1$ distinct p-cycles in $F_c(\underline{\underline{A}}_p) \cong C_p^c$, we get in all

$$\frac{p^c-1}{p-1} \cdot p^{c(c-1)}$$

(not necessarily independent) equations whose solutions are precisely the admissible values for $e_1,\dots,e_s$. Now

$$s = p^{c^2} > \frac{p^c-1}{p-1} \cdot p^{c(c-1)}$$

for all p and therefore there exist non-trivial solutions. This completes the proof.

<u>Proof of 8.1.</u>

We now prove $\ell(\underline{\underline{N}}\,\underline{\underline{A}}_n) \geq c$ where n is once again arbitrary (subject to $(m,n) = 1$).

Suppose 8.1 is false and that $F_{c-1}(\underline{\underline{N}}\,\underline{\underline{A}}_n)$ generates

$\underline{\underline{N}}\,\underline{\underline{A}}_n$. Then by Corollary 8.4, there exists a finite set Σ say, of normal subgroups of W_c with trivial intersection and with factor groups embeddable in $F_{c-1}(\underline{\underline{N}}\,\underline{\underline{A}}_n)$. Let p be any prime dividing n , and let P be the unique subgroup of the top group $F_c(\underline{\underline{A}}_n)$ of W_c , such that $P \cong C_p^c$. At least one normal subgroup say $M \in \Sigma$, must intersect P trivially. For otherwise Lemma 8.5 together with Lemma 7.4 would tell us that $\bigcap_{K \in \Sigma} K \neq E$. Thus W_c/M contains a subgroup isomorphic to C_p^c and therefore, by the supposition, so does $F_{c-1}(\underline{\underline{N}}\,\underline{\underline{A}}_n)$. If $F_{c-1}(\underline{\underline{N}}\,\underline{\underline{A}}_n)$ contains a subgroup $L \cong C_p^c$, then, by 8.8.3, L is contained in a subgroup isomorphic to C_n^{c-1} , which is impossible. We have reached a contradiction and the proof is complete.

BIBLIOGRAPHY

[1] Gilbert Baumslag, B.H. Neumann, Hanna Neumann and Peter M. Neumann, "On varieties generated by a finitely generated group". Math. Zeitschr. 86, 93-122 (1964).

[2] C.W. Curtis and I. Reiner, "Representation theory of finite groups and associative algebras". Interscience Publishers, New York, London (1962).

[3] M.J. Dunwoody, "On verbal subgroups of free groups". Arch. Math. 16, 153-157 (1964).

[4] Wolfgang Gaschütz, "Über die Φ-Untergruppe endlicher Gruppen". Math. Zeitschr. 58, 160-170 (1953).

[5] Marshall Hall Jr., "The theory of groups". Macmillan, New York (1959).

[6] P. Hall, "Some constructions for locally finite groups". J. London Math. Soc. 34, 305-319 (1959).

[7] Graham Higman, "Identical relations in finite groups". Conv. Internaz. di Teoria dei Gruppi Finiti (Firenze, 1960), 93-100. Edizioni Cremonese, Rome, 1960.

[8] Graham Higman, "Some ~~results~~ remarks on varieties of groups". Quart. J. Math. Oxford Ser. (2) 10, 165-178 (1959).

[9] Graham Higman, B.H. Neumann and Hanna Neumann, "Embedding theorems for groups". J. London Math. Soc. 24, 247-254 (1949).

[10] L.G. Kovács and M.F. Newman, "Cross varieties of groups". Submitted to Proc. Roy. Soc.

[11] L.G. Kovács and M.F. Newman, "On critical groups". J. Austral. Math. Soc. In Print.

[12] L. Kaloujnine and Marc Krasner, "Produit complet des groupes de permutations et problème d'extension des groupes. II". Acta. Sci. Math., Szeged 14, 39-66 (1951).

[13] A.G. Kurosh, "The theory of groups", Vols. 1 and 2. (English translation from the second Russian edition) Chelsea, New York (1956).

[14] S. Moran, "Associative operations on groups. I". Proc. London Math. Soc. Ser. 3, 6, 581-596 (1956).

[15] S. Moran, "The homomorphic image of the intersection of a verbal subgroup and the cartesian subgroup of a free product". J. London Math. Soc., 33, 237-245 (1958).

[16] B.H. Neumann, "Identical relations in groups. I". Math. Ann., 114, 506-525 (1937).

[17] B.H. Neumann, Hanna Neumann and Peter M. Neumann, "Wreath products and varieties of groups". Math. Zeitschr., 80, 44-62 (1962).

[18] Peter M. Neumann, "On the structure of standard wreath products of groups". Math. Zeitschr., 84, 343-373 (1964).

[19] M.F. Newman, "On a class of metabelian groups". Proc. London Math. Soc., Ser. (3), 10, 354-364 (1960).

[20] Sheila Oates and M.B. Powell, "Identical relations in finite groups". J. Algebra, 1, 11-39 (1964).

[21] A.L. Šmel'kin, "Wreath products and varieties of groups".. Ivz. Nauk S.S.S.R. Ser Mat., 29, 149-170 (1965). (Also for summary in English: Soviet Mathematics, Vol. 5, No. 4 (1964). Translation of Dokl. Akad. Nauk S.S.S.R. for Amer. Math. Soc.)

[22] Paul M. Weichsel, "A decomposition theory for finite groups with applications to p-groups". Trans. Amer. Math. Soc., 102, 218-226 (1962).